GEOMETRIC CONSTRUCTIONS APPROXIMATING π RELATED TO VIETA'S PRODUCT OF NESTED RADICALS

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1. Introduction

One of the three classical problems of antiquity is to "square the circle" [1]. Given a circle of radius *r*, construct, using only straightedge and compass, the side *x* of a square having same area. (Note that the straightedge must be unmarked.) Thus, to square the circle, one must construct the length (number) $\sqrt{\pi}\ r$. It is known that the only possible constructions are formed by adding, subtracting, multiplying, dividing and taking the square root of previously found lengths. These constructions may only be performed a finite number of times. It is impossible to construct π, and thus the squaring of the circle is impossible.

It is the purpose of this article to show how to construct geometrically approximations to π of arbitrary accuracy related to Vieta's famous infinite product for π :

(1) $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots.$$

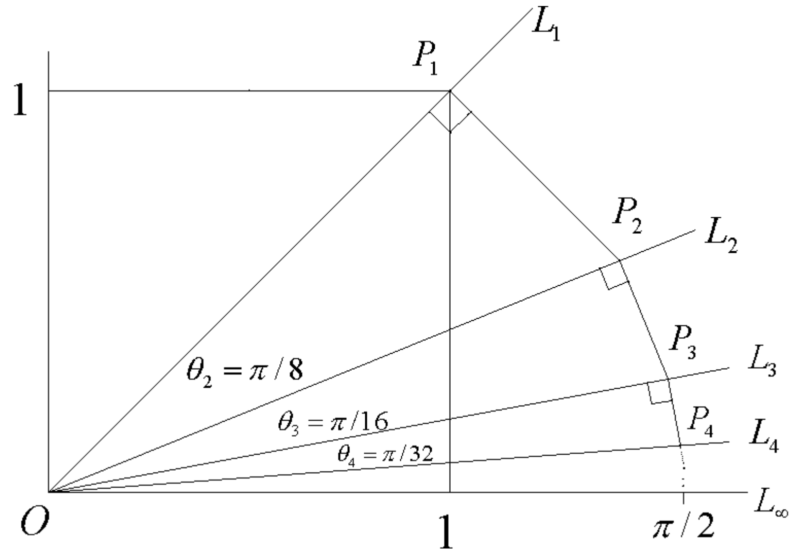

Figure 1: Constructing approximations to π/2

In Figure 1 we see the unit square. Angle L_1OL_∞ is $\pi/4$. Line OL_2 bisects angle L_1OL_∞ so that $\theta_2 = \pi/8$. Construct line OL_3 so that it bisects angle L_2OL_∞ making angle $\theta_3 = \pi/16$. Continuing we construct line OL_4 so that it bisects angle L_3OL_∞ making angle $\theta_4 = \pi/32$, etc.

From the corner of our unit square P_1 we construct a line perpendicular to line OL_1 meeting line OL_2 at P_2. From P_2 we construct a line perpendicular to line OL_2 meeting line OL_3 at P_3. We continue in this way forming additional points P_4, P_5, etc. The lengths of the line segments OP_1, OP_2, OP_3, … converge to $\pi/2$. In fact we will show that

$$\overline{OP_1} = \frac{1}{\sqrt{\frac{1}{2}}}, \quad \overline{OP_2} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}, \quad \overline{OP_3} = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}}, \ldots .$$

Comparing these with (1) we see that the length of $\overline{OP_n}$ is the reciprocal of the first *n* factors of Vieta's product.

We now review a derivation of Vieta's product (1). Repeated use of a familiar trigonometric identity gives us

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2},$$

$$\sin x = 2^2\cos\frac{x}{2}\cos\frac{x}{2^2}\sin\frac{x}{2^2},$$

$$\sin x = 2^3\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\sin\frac{x}{2^3}.$$

Continuing this way and dividing by x we get

(2) $$\frac{\sin x}{x} = \frac{\sin\frac{x}{2^N}}{x/2^N}\prod_{k=1}^{N}\cos\frac{x}{2^k},$$

or taking the reciprocal we get

(3) $$\frac{x}{\sin x} = \frac{x/2^N}{\sin\frac{x}{2^N}}\prod_{k=1}^{N}\sec\frac{x}{2^k}.$$

Next we use a half-angle formula to replace $\cos\frac{x}{2^k}$:

$$\cos\frac{x}{2} = \sqrt{\frac{1}{2}+\frac{1}{2}\cos x},$$

$$\cos\frac{x}{2^2} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}},$$

$$\cos\frac{x}{2^3} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}}}.$$

Now (2) becomes

$$\frac{\sin x}{x} = \frac{\sin\frac{x}{2^N}}{x/2^N}\prod_{k=1}^{N}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}}}}}_{k\ radicals}.$$

Finally we set $x = \pi / 2$ and let N tend to infinity to get Vieta's product (1). Notice that (3) now takes the form

(4) $$\frac{\pi}{2} = \prod_{k=1}^{\infty} \sec \frac{\pi}{2^{k+1}}.$$

From Figure 1 we see that

$$\overline{OP_1} = \sec(\pi / 4),$$

$$\overline{OP_2} = \overline{OP_1} \sec(\pi / 8) = \sec(\pi / 4)\sec(\pi / 8),$$

$$\overline{OP_3} = \overline{OP_2} \sec(\pi / 16) = \sec(\pi / 4)\sec(\pi / 8)\sec(\pi / 16)$$

etc. It is now clear from (4) that our constructions converge to $\pi / 2$. Even though our constructions approach $\pi /2$ as a limiting case, we cannot say that we have constructed $\pi /2$ since limiting cases are not permitted in classical geometric construction.

The book [2] is a classic and is now available in an inexpensive paperback edition. A very readable discussion of constructible numbers, with complete proofs that require only a precalculus background, is found in chapter 3. It is strongly recommended. New papers generalizing Vieta's product are [3, 4, 5 and 6].

References

[1] Burton, D. M. (1999), *The History of Mathematics, An Introduction,* (4th Ed.), WCB/McGraw-Hill, New York, pp. 115-122.

[2] Courant, R. and Robbins, H. (1996), (revised by Stewart, I.), *What is Mathematics?*, (2nd Ed.), Oxford University Press, New York.

[3] Osler, T. J., *The united Vieta's and Wallis's products for pi*, American Mathematical Monthly, 106 (1999), pp. 774-776.

[4] Osler, T. J. and Wilhelm, M., *Variations on Vieta's and Wallis's products for pi*, Mathematics and Computer Education, 35(2001), pp. 225-232.

[5] Osler, T. J., *An unusual product for sin z and variations of Wallis's product*, The Mathematical Gazette, 87(2003), pp. 134-139.

[6] Osler, T. J., *The general Vieta-Wallis product for pi*, to appear in The Mathematical Gazette (November, 2005.)