A REMARKABLE FORMULA FOR APPROXIMATING THE SUM
OF ALTERNATING SERIES

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1. Introduction

In this paper we present a simple formula that can often be used to assist in the
approximation of the sum of an alternating series

\[ \sum_{n=1}^{\infty} (-1)^{n+1} f(n). \]

The ideas presented here were motivated by translating Euler’s papers [1] and [2].

Suppose we begin by summing \( a \) terms of this series exactly, then the approximation that
we will obtain is given by

\[ \sum_{n=1}^{\infty} (-1)^{n+1} f(n) \approx \sum_{n=1}^{a} (-1)^{n+1} f(n) + \frac{7f(a+1) + f(a+2) - 2f(a+3)}{12}. \]

Notice that we are adding the simple correction term

\[ A(a+1) = \frac{7f(a+1) + f(a+2) - 2f(a+3)}{12} \]

to account for the infinite number of terms that have been neglected. It is remarkable that
that (3) contains no integrals or derivatives. It is an algebraic linear sum of the three
consecutive terms starting at \( n = a + 1 \).

This paper is not rigorous. We present a simple informal derivation of (3) in the
next section, but no estimate of the error is attempted. Rather our purpose is to show
some remarkable examples of the use of (3) with a series closely related to the zeta
function. The most surprising examples occur when the series diverges.

2. Informal derivation of the summation formula

Beginning with the Euler-Maclaurin summation formula (see [3]),

\[
\sum_{n=0}^{N} f(a + nh) = \frac{1}{h} \int_{a}^{b} f(x) dx + \frac{f(b) + f(a)}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} \left( f^{(k-1)}(b) - f^{(k-1)}(a) \right) h^{k-1}.
\]

Here \( b = a + hN \) and the numbers \( B_k \) are the well known Bernoulli numbers. The first
term in the summation is \( \frac{B_2}{2!} (f'(b) - f'(a)) h \), and since \( B_2 = \frac{1}{6} \) this term becomes

\[
\frac{f'(b) - f'(a)}{12} h.
\]

Approximating the derivatives by \( f'(a) \approx \frac{f(a+h) - f(a)}{h} \) and

\[
f'(b) \approx \frac{f(b+h) - f(b)}{h}
\]

we get

\[
\sum_{n=0}^{N} f(a + nh) \approx \frac{1}{h} \int_{a}^{b} f(x) dx + \frac{f(b) + f(a)}{2} + \frac{f(a) - f(a+h)}{12} - \frac{f(b) - f(b+h)}{12},
\]

where we have dropped terms with powers of \( h \) greater than 1. (The above formula was
first found by Euler in [1] with a different derivation.) Letting \( N \to \infty \) we get

\[
\sum_{n=0}^{\infty} f(a + nh) \approx \frac{1}{h} \int_{a}^{b} f(x) dx + \frac{7f(a) - f(a+h)}{12}.
\]

We now employ an idea used by Euler in [2]. To obtain the alternating signs, in place of
\( h \) let us write \( 2h \) to get the summation:

\[
\sum_{n=0}^{\infty} f(a + 2nh) \approx \frac{1}{2h} \int_{a}^{b} f(x) dx + \frac{7f(a) - f(a+2h)}{12}.
\]

Double this and subtract the preceding series from it to get the alternating series version
of Euler’s summation formula
\[ \sum_{n=0}^{\infty} (-1)^n f(a + nh) \approx \frac{7f(a) + f(a + h) - 2f(a + 2h)}{12}. \]

Set \( h = 1 \) and get

\[ \sum_{n=a}^{\infty} (-1)^n f(n) \approx \frac{7f(a) + f(a+1) - 2f(a+2)}{12}. \]

Notice that this result is particularly simple since the integral has disappeared.

### 3. Examples of summing alternating series

We will now give examples of approximating the sum of alternating series

\[ \sum_{n=1}^{\infty} (-1)^{n+1} f(n) \]. In all cases we will begin by finding the exact sum of the first \( a \) terms

\[ \sum_{n=1}^{a} (-1)^{n+1} f(n) \], then add an approximation of the remaining terms from (3) which we call

\[ A(a+1) = \frac{7f(a+1) + f(a+2) - 2f(a+3)}{12}. \]

Throughout the paper, numerical and graphical results were obtained using Mathematica.

#### Example 1
Let \( f(x) = \frac{1}{x} \) and consider the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \). We obtain the results shown in the following table:

<table>
<thead>
<tr>
<th>( a ) = number of terms in the exact part of the sum</th>
<th>Exact finite series</th>
<th>Addition of the correction term</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.64563492063492063492</td>
<td>0.69278915528915528916</td>
</tr>
<tr>
<td>100</td>
<td>0.68817217931019520324</td>
<td>0.69314662735465698809</td>
</tr>
<tr>
<td>1000</td>
<td>0.69264743055982030967</td>
<td>0.69314717997972543014</td>
</tr>
<tr>
<td>10000</td>
<td>0.69309718305994529692</td>
<td>0.69314718055936228847</td>
</tr>
<tr>
<td>100000</td>
<td>0.69314218058494530942</td>
<td>0.69314718055994472612</td>
</tr>
</tbody>
</table>
We know that this series sums to \( \log 2 = 0.69314718055994530942 \), so the last result in the above table is accurate to 14 decimal places. While 100,000 terms of the exact series yields 5 correct decimal places, the addition of the correction term \( A(100,001) \) adds an additional 9 decimal places!

The remaining examples are concerned with the zeta function which is usually defined by the series

\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.
\]

This series is valid for \( \text{Re}(z) > 1 \). Using elementary algebra it is easy to show that we can also use an alternating series \( \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \) to define the zeta function as

\[
\zeta(z) = (1 - 2^{1-z})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = (1 - 2^{1-z})^{-1} \eta(z).
\]

Because this series is alternating, it converges in the wider range \( \text{Re}(z) > 0 \).

**Example 2** In this example we examine the series \( \eta(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \). We find that

\[
\sum_{n=1}^{10000} \frac{(-1)^{n+1}}{n^2} = 0.82246702842461321823
\]

\[
\eta(2) \approx \sum_{n=1}^{10000} \frac{(-1)^{n+1}}{n^2} + A(10001) = 0.82246703342411304336
\]

where \( A(10001) \) is calculated from (3) with \( f(x) = \frac{1}{x^2} \). The exact value of \( \eta(2) \) is
0.8224670334241321824 to 20 decimal places. We see that the exact finite sum gives us 8 correct decimal places, while the addition of the correction term (3) adds an additional 8 correct decimal places.

Example 3 In this example we examine the Leibniz series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \pi \). We obtain the results shown in the following table:

<table>
<thead>
<tr>
<th>( a ) = number of terms in the exact part of the sum</th>
<th>Exact finite series</th>
<th>Addition of the correction term</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{n=1}^{a} (-1)^{n+1} \frac{4}{2n-1} )</td>
<td>( \sum_{n=1}^{a} (-1)^{n+1} \frac{4}{2n-1} + A(a+1) )</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>3.0418396189294022111</td>
<td>3.1407768169970350614</td>
</tr>
<tr>
<td>100</td>
<td>3.1315929035585527643</td>
<td>3.1415915307119443629</td>
</tr>
<tr>
<td>1000</td>
<td>3.1405926538397929260</td>
<td>3.1415926524276141863</td>
</tr>
<tr>
<td>10000</td>
<td>3.1414926535900432385</td>
<td>3.141592653886270217</td>
</tr>
<tr>
<td>100000</td>
<td>3.1415826535897934885</td>
<td>3.1415926535897920718</td>
</tr>
</tbody>
</table>

The value of \( \pi \) correct to 20 digits is 3.1415926535897932385. We see that the addition of our three term correction more than doubles the number of correct digits.

4. Examples of divergent alternating series

The series \( \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \) can be used to compute the zeta function from \( \zeta(z) = \left(1 - 2^{-z}\right)^{-1} \eta(z) \). However this series only converges in the right half plane \( 0 < \text{Re}(z) \). To find values of the zeta function in \( \text{Re}(z) \leq 0 \) we usually employ the functional equation
\[
\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(s).
\]

Surprisingly, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \) can be used to roughly estimate values of the zeta function even in the region where it diverges provided we are not too far away from the imaginary axis. We will demonstrate this in the following examples. Again, we will use the approximation

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \approx \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^z} + A(a+1),
\]

where from (3) we have

\[
A(a+1) = \frac{1}{12} \left( \frac{7}{(a+1)^2} + \frac{1}{(a+2)^2} - \frac{2}{(a+3)^2} \right).
\]

Example 4 We estimate \( \eta(-1/2) \) using the divergent series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{-1/2}} \). The results of our computations are shown in the following table:

<table>
<thead>
<tr>
<th>a, number of terms in the exact part of the sum</th>
<th>Exact finite series ( \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^{-1/2}} )</th>
<th>Addition of the correction term ( \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^{-1/2}} + A(a+1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-1.2405379939375849962</td>
<td><strong>0.3819072453879582</strong></td>
</tr>
<tr>
<td>100</td>
<td>-4.6323951092721502095</td>
<td><strong>0.380175820396505699</strong></td>
</tr>
<tr>
<td>1000</td>
<td>-15.43523633506370391</td>
<td><strong>0.38010711275097133</strong></td>
</tr>
<tr>
<td>10000</td>
<td>-49.621145187389534733</td>
<td><strong>0.380104885506823957</strong></td>
</tr>
<tr>
<td>100000</td>
<td>-157.73417348051680116</td>
<td><strong>0.38010481491544972</strong></td>
</tr>
</tbody>
</table>

The exact value is \( \eta(-1/2) = 0.38010481260968401678 \) to 20 decimal places. In this case the exact finite series gives no indication of the correct numerical value. Yet with the
addition of the correction term (4) we were able to get 8 correct decimal places from this divergent series. This looks to us like a miracle!

**Example 5** We can also calculate with complex values. We estimate \( \eta(-1/2 + i) \) using the divergent series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{-1/2+i}} \). The results are shown in the following table:

<table>
<thead>
<tr>
<th>(a) = number of terms in the exact part of the sum</th>
<th>Exact finite series ( \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^{-1/2+i}} )</th>
<th>Addition of the correction term ( \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^{-1/2+i}} + A(a+1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.545836 + 1.413568i</td>
<td>0.3970736 + 0.2551487i</td>
</tr>
<tr>
<td>100</td>
<td>0.915657 - 4.725596i</td>
<td>0.4040773 + 0.2611489i</td>
</tr>
<tr>
<td>1000</td>
<td>-12.42090 + 9.51521i</td>
<td>0.4041184 + 0.2607885i</td>
</tr>
<tr>
<td>10000</td>
<td>-49.26067 + 10.89852i</td>
<td>0.4041087 + 0.2607952i</td>
</tr>
</tbody>
</table>

The exact value is

\[ \eta(-1/2 + i) = 0.4041091417758785008+0.26079531201772776628i \]

to 20 decimal places. As in the previous example, the exact finite sum gives no indication of the correct numerical value, yet 5 correct decimal places were obtained by the addition of the correction term (4).

**Example 6** We estimate \( \eta(-3/2) \) using the divergent series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{-3/2}} \). The results are shown in the following table:

<table>
<thead>
<tr>
<th>(a) = number of terms in the exact</th>
<th>Exact finite series ( \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^{-3/2}} )</th>
<th>Addition of the correction term ( \sum_{n=1}^{a} \frac{(-1)^{n+1}}{n^{-3/2}} + A(a+1) )</th>
</tr>
</thead>
</table>
The exact value is \( \eta(-3/2) = 0.11868087071984021204 \) to 20 decimal places.

Apparently we are near the limits of this method.

**Example 7.** In this final example we consider using the divergent series

\[
\sum_{n=1}^{\infty} (-1)^{n+1} n^{-3/2 - y i}
\]

to estimate the value of \( |\eta(-1 + iy)| \) for \( 0 \leq y \leq 50 \). In figure 1 we see \( y \) plotted as the horizontal axis with the sum of the first 100 terms of the series plotted vertically.

**Figure 1: The sum of 100 terms of the exact series**

In Figure 2 we add the correction term \( A(101) \) to the series shown in the previous figure.

We compare this with \( \eta(-1 + y i) \).
Figure 2: Comparing the exact value of $|\eta(-1 + iy)|$ with the corrected series

$$\sum_{n=1}^{100} (-1)^{n+1} n^{1-iy} + A(101).$$

The graphs show that the addition of the simple correction term

$$A(101) = \frac{1}{12} \left( \frac{7}{101^{1+yi}} + \frac{1}{102^{1+yi}} - \frac{2}{103^{1+yi}} \right)$$

yields values that are surprisingly close to $|\eta(-1 + iy)|$.

References
