ABSTRACT

LEIBNIZ RULE, THE CHAIN RULE, AND TAYLOR'S THEOREM FOR FRACTIONAL DERIVATIVES

by Thomas J. Osler

Research Advisor: Prof. Samuel Karp

A fractional differential operator is an extension of the familiar operator $D^n = d^n/dz^n$ in which $n$ can assume non-integral values. In this thesis three elementary formulas from the calculus are generalized to include fractional derivatives: (1) the Leibniz rule for the derivative of the product of two functions, (2) the chain rule for the derivative of a composite function, and (3) Taylor’s theorem for the expansion of a function in a power series. Each of the three generalized results is applied to the investigation of the special functions of mathematical physics.

Chapter I begins with a brief historical survey of previous attempts to generalize Leibniz rule to include fractional derivatives. Classical methods of defining the fractional derivative are reviewed. A new definition of the fractional derivative of order $\alpha$ with respect to an arbitrary function $g(z)$ denoted by $D_{g(z)}^\alpha f(z)$ is given. While this new definition is only a slight general-
ization of certain special cases which have appeared in the literature, it makes possible certain manipulations which were not considered previously.

A brief list of special functions of mathematical physics expressed as fractional derivatives using this new notation is given. This list reveals that known representations for higher functions in the form of derivatives of elementary functions (like the Rodrigues formula for the Legendre polynomials) can be generalized through fractional derivatives to include a much wider family of functions.

Next a new proof for a Leibniz rule for the fractional derivative of the product of two functions is given. Here $D^\alpha_g(z) u(z)v(z)$ is expanded in an infinite series whose terms involve products of generalized binomial coefficients with fractional derivatives of $u(z)$ and $v(z)$. The region in the $z$-plane in which this series converges is revealed for the first time.

A new series expansion for $D^\alpha_g(z)f(z,z)$ which is a generalization of the above Leibniz rule is derived.

Chapter I ends with an examination of special cases of the generalized Leibniz rule. These reveal several new and some known series expansions relating the special functions of mathematical physics.

In Chapter II the formula from the elementary
calculus for the n-th derivative of a composite function is generalized to arbitrary complex n. A fundamental relation expressing derivatives with respect to g(z) in terms of derivatives with respect to h(z) is derived.

Using the Leibniz rule for the fractional derivative discussed in Chapter I, an infinite series expansion is derived which proves to be a generalization of the chain rule familiar from the elementary calculus.

Examples of these formulas are examined. Novel derivations of the known results expressing the hypergeometric function of argument -1, and of argument 1/2 in terms of the gamma function are given, as well as new results.

In Chapter III a generalization of Taylor's theorem to power series with non-integral exponents is examined. In 1945, G. H. Hardy gave a derivation of this formula (known as the Taylor-Riemann theorem) using real variable methods. A new proof for the Taylor-Riemann theorem is given using a Cauchy integral formula for fractional derivatives.

Finally, the Taylor-Riemann theorem is used as a basis for a new approach to the formal discovery of a generating function for a given specific family of special functions. Again the representation of the higher special functions in terms of fractional derivatives of the elementary functions is used to advantage.
LEIBNIZ RULE, THE CHAIN RULE, AND TAYLOR’S THEOREM
FOR FRACTIONAL DERIVATIVES

by Thomas J. Osler
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REFERENCES
INTRODUCTION

A fractional differential operator is an extension of the familiar operator $D^n = d^n/dz^n$ in which $n$ can assume non-integral values. While the subject of fractional differentiation is as old as the calculus itself (dating back to Leibniz [14]), a thorough investigation of even the most basic properties of fractional derivatives remains to be completed. In particular, many formulas and techniques familiar from the calculus of ordinary derivatives and integrals should be fully generalized to derivatives of arbitrary order to complete the fractional calculus. This paper is directed toward that end. Here three elementary formulas from the calculus are generalized to include fractional derivatives: (1) the Leibniz rule for the derivative of the product of two functions, (2) the chain rule for the derivative of a composite function, and (3) Taylor's theorem for the expansion of a function in a power series. Each of the three generalized results is applied to the study of the special functions of mathematical physics. New relations among these functions, and simpler derivations of known results are obtained.

Historically, four early approaches to the problem of finding a meaningful definition of a differential
operator of arbitrary order are worthy of note. First Liouville [16] in 1832, examined the formula $D^n e^{az} = a^n e^{az}$, and allowed $n$ to assume fractional values. He then expanded an arbitrary function in its Fourier series and obtained a fractional calculus by differentiating fractionally termwise. Riemann [19] initiated a second investigation in an unpublished paper by considering power series with non-integral exponents. Through an examination of the coefficients of these powers he was led to an expression for a fractional derivative which did not agree with Liouville's results. In 1868, a third study by Grunwald [8] was based on a direct generalization of the finite difference quotient by which ordinary derivatives are defined. Grunwald then obtained definite integral representations of fractional derivatives through which Riemann's and Liouville's results were coordinated. A fourth investigation by Nekrassov [17] in 1888 was based on Cauchy's integral formula, and introduced the powerful methods of complex variable theory into this study.

The recent literature [4, 5, 11, 13, 20] includes many illustrations of the application of the fractional calculus to the solution of problems in ordinary, partial,
and integral equations. These papers demonstrate the remarkably suggestive nature of the fractional derivative operator. Higgins [13] has observed that "although results using fractional integral operators can always be obtained by other methods, the succinct simplicity of the formulation may often suggest approaches which might not be evident in a classical approach."

The author rediscovered the definitions and many of the basic properties of fractional derivatives during the summer of 1968. He invented a definition and notation for the fractional derivative with respect to an arbitrary function \( g(z) \), denoted by \( D^\alpha_{g(z)} f(z) \). While this is only a slight extension of the familiar definition found in the literature, it makes possible an understanding of a generalized chain rule, and a clearer view of other results in the fractional calculus.

Guided by his advisor, Professor S. N. Karp of New York University, the author determined that significant
areas of a calculus for fractional derivatives had been neglected in the literature. The synopsis enumerates the new results presented in this paper.

It is the author's hope that this paper will reveal some of the advantages of using fractional differential operators as a tool for discovery in analysis and applied mathematics.
SYNOPSIS OF THE THESIS AND SUMMARY OF NEW RESULTS

In the synopsis which follows, original results and significant new interpretations of known results have been indented and numbered for ease of reference.

CHAPTER I:

Chapter I begins with a brief survey of previous attempts to generalize Leibniz rule to include fractional derivatives. Classical methods of defining the fractional derivative are reviewed.

1. A new definition of the fractional derivative of order $\alpha$ with respect to an arbitrary function $g(z)$, denoted by $D_{g(z)}^\alpha f(z)$ is given in Definition 3.1, page 22.

While this new definition is only a slight generalization of certain special cases which have appeared in the literature $[4,5]$, it makes possible certain manipulations which were not considered before.

2. A brief list of special functions of
mathematical physics expressed as fractional derivatives using this new notation is given in Table 6.1, page 42.

Table 6.1 reveals that known representations for higher functions in the form of derivatives of elementary functions (like the Rodrigues formula for the Legendre polynomials) can be generalized through fractional derivatives to include a much wider family of functions.

Next the Leibniz rule for the fractional derivative of the product of two functions

\[ (0.1) \ D_{g(z)}^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} (\gamma + n) D_{g(z)}^{\alpha-\gamma-n} u(z) D_{g(z)}^{\gamma+n} v(z) \]

is examined. Here \( \gamma \) is an arbitrary complex number.

3. A new proof of this Leibniz rule (0.1) is given in Theorem 4.1, page 28. The region in the z-plane in which the series (0.1) converges is revealed for the first time.
In 1931, Watanabe [21] discussed (0.1), but he gave the wrong region of convergence. Watanabe's Leibniz rule (0.1) extends in a natural way to the more general result

\[ (0.2) \quad D^\infty g(z) f(z,z) = \sum_{n=-\infty}^{\infty} \left( \begin{array}{c} \infty \\ -\gamma \end{array} \right)^{\infty-n, \gamma+n} D^\infty g(z), g(w) \left. f(z,w) \right|_{w=z}. \]

4. The new generalization of Leibniz rule (0.2) is derived in Theorem 5.1, page 38.
The special case of (0.2) in which \( f(z,w) = u(z)v(w) \) is Watanabe's Leibniz rule (0.1).

Chapter I ends with an examination of twenty-one special cases of the generalization of Leibniz rule (0.2). These examples are constructed with the aid of the fractional derivative representations of the special functions listed in Table 6.1.

5. Table 6.3, pages 45 - 48, lists several new and some known infinite series expansions relating the special functions of mathematical physics. These are special cases of the generalization of Leibniz rule (0.2).
It is remarkable that the standard derivations of series rediscovered by this method are more difficult that the derivation of the Leibniz rule itself.

CHAPTER II:

In Chapter II the formula from the elementary calculus for the n-th derivative of a composite function is generalized to arbitrary non-integral n.

6. The fundamental relation

\[ (0.3) \quad D_{g(z)}^\alpha f(z) = D_{h(z)}^\alpha \left\{ \frac{f(z)g'(z)}{h'(z)} \left( \frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\alpha+1} \right\} \bigg|_{w=z} \]

relating derivatives with respect to g(z) to derivatives with respect to h(z) is derived in Theorem 8.1, page 52.

The Leibniz rule for fractional derivatives (0.1) is applied to the fundamental relation (0.3) to obtain the expansion
\begin{equation}
(0.4) \quad D_{g(z)}^{\alpha} f(z) = \sum_{h=-\infty}^{\infty} \left( D_{h(z)}^{\gamma+n} f(z) \right) \frac{F(z,w)}{F(z,w)} \times \left\{ \begin{array}{c}
F(z,w)g'(z) \\
h'(z)
\end{array} \right\}
\left( \frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\alpha+1} \bigg|_{w=z}
\end{equation}

Here $\gamma$ is an arbitrary complex number and $F(z,w)$ is an arbitrary function.

7. The generalized chain rule (0.4) is derived in Theorem 8.2, page 54. The special case in which $\gamma = 0$, $F(z,w) \equiv 1$, and $\alpha$ is a natural number is a known result in the calculus. If $\alpha = 1$ we have the familiar chain rule.

Chapter II closes with an examination of a few special cases of the generalized chain rule.

8. Examples of the formulas (0.3) and (0.4) are examined in Section 9, pages 56 - 61, for specific functions. Novel derivations of the known results expressing the hypergeometric function of argument $-1$, and of argument $1/2$
in terms of the gamma function are given, as well as new results.

CHAPTER III:

In Chapter III the generalization of Taylor's theorem to power series with non-integral exponents,

$$f(z) = \sum_{n=-\infty}^{\infty} \left. \frac{\gamma^{+n}}{D_{z-b} f(z)} \right|_{z=a} \frac{\gamma^{+n}}{(z-a)} \frac{1}{\Gamma(\gamma^{+n+1})}$$

(Taylor - Riemann theorem) is examined. The special case of (0.5) in which \( \gamma = 0 \) is recognized at once as the elementary power series expansion of \( f(z) \) about \( z = a \).

Riemann [19] gave the formula (0.5) formally in order to generate a definition of fractional derivative. Riemann's analysis was strictly formal, and he never considered the convergence of the series. Heaviside [10] gave a formal discussion of the two special cases of (0.5) in which \( f(z) = z^p \) and \( f(z) = e^{cz} \). The first rigorous examination of the convergence of (0.5) was given by G. H. Hardy [9] in 1945, when he considered (0.5) as an asymptotic expansion of \( f \), and as a series summable (Borel) to \( f \). Hardy restricts \( \gamma, a, \) and \( z \) to be real, and considers only the two cases in which \( b = 0 \), and \( b = -\infty \). The question of the pointwise convergence of the series (0.5) to the function \( f(z) \) for real or complex \( z \) has, to the best of the author's knowledge, never been considered in the literature.
9. The ordinary pointwise convergence of the series (0.5) to the function \( f(z) \) in the complex z-plane is considered for the first time in Theorems 11.1 and 11.2, pages 65-67, for arbitrary complex \( \gamma \), \( a \), and \( b \). The region of convergence of the series in the z-plane for suitable functions \( f(z) \) is stated.

Finally, the Taylor - Riemann theorem is used to discover generating functions for certain families of special functions. Again the representation of the higher special functions in terms of fractional derivatives of the elementary functions is used to advantage (Table 13.1, page 77).

10. A new systematic approach to the formal discovery of a generating function for a specific given family of special functions is discussed in Section 12. Examples of the method are given in Section 13, pages 72 - 76, revealing both old and new generating functions. In the case of known generating functions, connections between hitherto seemingly unrelated results are revealed.
CHAPTER I
FRAGMENTAL DERIVATIVES AND LEIBNIZ RULE GENERALIZED

1. Introduction
In this chapter certain generalizations of the Leibniz rule for the derivative of the product of two functions are examined and used to generate several infinite series expansions relating special functions. We first review various definitions which have been proposed to generalize the order of the differential operator \( D^n_z \) (\( = d^n/dz^n \)), considering finally a derivative of arbitrary order \( \alpha \) with respect to \( g(z) \) of \( f(z) \) which we denote by the symbol \( D^\alpha_{g(z)} f(z) \). The latter reduces to the usual differential operator when \( \alpha = 0,1,2,\ldots \) and \( g(z) = z \). A new proof for the formula

\[
(1.1) \quad D^\alpha_{g(z)} u(z) v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D^{\alpha-\gamma-n}_{g(z)} u(z) D^{\gamma+n}_{g(z)} v(z),
\]

where \( \binom{\alpha}{\gamma+n} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \gamma - n + 1) \Gamma(\gamma + n + 1)} \).
for the fractional derivative of the product \( uv \) is given which reveals for the first time the precise region of convergence of the series in the \( z \)-plane. The special case of (1.1) in which \( \gamma' = 0 \), and \( \alpha = 0, 1, 2, \ldots \) is the Leibniz rule from the elementary calculus. The formula

\[
(1.2) \quad D_\gamma^\alpha g(z) f(z, z) = \sum_{n=-\infty}^{\infty} \left( \gamma + n \right)^{\alpha - \gamma - n, \gamma + n} D_\gamma^\alpha g(z), g(w) f(z, w) \bigg|_{w=z}
\]

is shown to be a natural outgrowth of the generalized Leibniz rule (1.1). Formula (1.1) is the special case of (1.2) in which \( f(z, w) = u(z)v(w) \). Formula (1.2) and several of the series expansions obtained from it appear to be new. Studies of a Leibniz rule for derivatives of arbitrary order date back to 1832 when Liouville [16,p.117] gave the special case of (1.1) in which \( \gamma' = 0 \)

\[
(1.3) \quad D_\gamma^\alpha u(z)v(z) = \sum_{n=0}^{\infty} \frac{D_\gamma^{\alpha - n} u(z) D_\gamma^n v(z)}{\Gamma(\alpha + 1) \frac{\Gamma(\alpha - n + 1) n!}{n!}}.
\]

Liouville invented a fractional derivative by observing that the simple relation \( D^n e^{az} = a^n e^{az}, n = 0, 1, 2, \ldots \),
could be generalized for arbitrary \( \alpha \) by \( D^\alpha e^{az} = a^\alpha e^{az} \).

Liouville then expanded a general function in a Fourier series and obtained its fractional derivative by differentiating term by term.

In 1867 and 1868 A. K. Grunwald [8, pp. 406-468] and A. V. Letnikov [15] found (1.3) by starting with a fractional derivative based on the Riemann-Liouville integral

\[
D^\alpha_z f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{\alpha+1}}
\]

A further extension of (1.3) was given by E. Post [18, p. 755] in 1930 to functions of \( D^\alpha \) more general than \( D^\alpha \). In 1961 M. A. Bassam [1] presented another derivation of (1.3), and recently M. Gaer and L. A. Rubel [6] have found a rule for the fractional derivative of a product which does not reduce to the Leibniz rule.

In 1931 Y. Watanabe [21] studied (1.3) and (1.1) by expanding \( u(z) \) and \( v(z) \) in power series in \( z \). His method does not yield the precise region of convergence in the \( z \)-plane. On the following pages a simpler proof of (1.1) is given using a Cauchy integral formula for fractional derivatives. The use of this powerful tool permits us to determine the precise region of convergence.
Although the Cauchy integral formula for fractional derivatives appeared as early as 1888 [17] it seems appropriate to include a brief section motivating this concept to make the paper self-contained.

The concept of fractional derivative with respect to an arbitrary function has been used by A. Erdelyi [4,5] . A natural notation is introduced for this concept which is useful and suggestive in applications, yet seems to be new. A very short table of special functions represented by fractional derivatives incorporating this notation is included.

Finally (1.2) is used to generate certain infinite series expansions relating special functions of mathematical physics by assigning specific values to the functions \( f \) and \( g \), and to the parameters \( \alpha \) and \( \gamma \). Some of the expansions thus generated are known, while others appear to be new. It is remarkable that the proof of (1.2) is so simple, that it is easier than the usual derivations of known expansions obtainable from it. An example is Dougall's formula [2, vol. 1, p.7]

\[
\frac{n^2 \Gamma(c+d-a-b-1) \csc \pi \alpha \csc \pi \beta}{\Gamma(c-a) \Gamma(c-b) \Gamma(d-a) \Gamma(d-b)} = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n) \Gamma(d+n)}
\]

which can be obtained from (1.1) immediately by taking \( u = z^{d-a-1}, \ v = z^{c-b-1}, \ g = z, \ \alpha = c-a-1, \) and \( \gamma = c-1. \)
2. Motivation of fractional derivatives

When motivating the concept of a derivative whose order is not to be the usual integer value, it is perhaps simplest to begin with the function $z^p$. The elementary formula, for $n$ and $p$ as natural numbers,

$$D^n_z z^p = \frac{p!}{(p-n)!} z^{p-n},$$

generalizes at once to

$$D^\alpha_z z^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta - \alpha + 1)} z^{\beta - \alpha},$$

where the only restriction is that $\beta \neq -1, -2, \ldots$ .

We can now define the fractional derivative of $z^\beta f(z)$ where $f(z)$ is analytic at $z = 0$, by differentiating the power series for $z^\beta f(z)$ term by term. We get

$$D^\alpha_z z^\beta f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{\Gamma(\beta + n + 1)}{\Gamma(\beta - \alpha + n + 1)} z^{\beta - \alpha + n}.$$

This series has the same circle of convergence as the power series for $f(z)$ about $z = 0$. 
The usual starting point for a definition of fractional derivative taken in recent papers [1,4,5,12] is the Riemann-Liouville fractional integral

$$(2.1) \quad D_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(\zeta) d\zeta}{(z-\zeta)^{\alpha+1}}.$$ 

Here the path of integration is along a line from 0 to z in the complex \(\zeta\)-plane, and \(\text{Re}(\alpha)<0\). This integral can be motivated from the Cauchy formula for a repeated integral

$$\int_0^z \int_0^{t_3} \int_0^{t_2} \cdots f(t_1) \, dt_1 \, dt_2 \, \cdots \, dt_n = \frac{1}{(n-1)!} \int_0^z \frac{f(t) dt}{(z-t)^{n+1}}.$$ 

Setting \(f(t_1) = t_1^\beta\) in this formula we see that the left hand side becomes

$$\frac{z^{\beta+n}}{(\beta+1)(\beta+2) \cdots (\beta+n)} = \frac{\Gamma(\beta+1)z^{\beta+n}}{\Gamma(\beta+n+1)} = D_z^{-n} z^\beta.$$
Formally replacing \(-n\) by \(\infty\) we get

\[
D_{\zeta}^\infty \zeta^\beta = \frac{1}{\Gamma(-\infty)} \int_0^{\zeta} \frac{t^\beta \, dt}{(\zeta-t)^{\infty+1}}
\]

Now the Riemann-Liouville integral seems somewhat reasonable as a definition for fractional derivative. In fact, the reader can quickly convince himself that this integral and the previous power series definition are equivalent using the elementary properties of the gamma and beta function.

A third method for motivating the concept of a derivative of arbitrary order is to examine Cauchy's integral formula from complex variables

\[
D_{\zeta}^n f(\zeta) = \frac{n!}{2\pi i} \oint_c \frac{f(\zeta)}{(\zeta - \zeta)^{n+1}} \, d\zeta
\]

If we replace \(n\) by non integer \(\infty\) in this formula, \((\zeta - \zeta)^{-\infty+1}\) no longer has a pole at \(\zeta = \zeta\), but a branch point. We are no longer free to deform the contour \(c\) at will, for the value of the integral is now a function of the point where \(c\) crosses the branch line for \((\zeta - \zeta)^{-\infty+1}\). Take this point as zero as shown in Figure 2.1.
Figure 2.1
If we deform the contour \( C \) to \( C' \) we see that

\[
\frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(\gamma) d \gamma}{(\gamma - z)^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{2\pi i} \left[ 1 - e^{-2\pi i (\alpha + 1)} \right] \int_0^z \frac{f(\gamma) d \gamma}{(\gamma - z)^{\alpha+1}}
\]

\[
= \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(\gamma) d \gamma}{(z - \gamma)^{\alpha+1}}.
\]

Since this is the Riemann Liouville integral we have the following generalized Cauchy integral formula.

\[
(2.2) \quad D_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_C \frac{f(\gamma) d \gamma}{(\gamma - z)^{\alpha+1}}.
\]

The significance of the fact that \( C \) must start and end at \( \gamma = 0 \) will appear in the following section. This formula was given as far back as 1888 by Nekrassov [17].

It is important to note that each of the three methods for defining a fractional derivative feature certain peculiar advantages and disadvantages. Using the power series method we can differentiate \( z^p \) so long as \( p \) is not a negative integer. However, for the Riemann Liouville integral, or the generalized Cauchy integral we require \( \text{Re}(p) > -1 \) so that the integrals are defined. The power series method fails
to differentiate functions whose singularities at \( z = 0 \) are not of the type \( z^\beta \). Such a function is \( \log z \) which can be handled by the other two methods.

The Riemann Liouville integral requires \( \text{Re}(\alpha) < 0 \), whereas the power series method has no restriction on \( \alpha \), and the Cauchy integral formula requires only that \( \alpha \) not be a negative integer. The generalized Cauchy integral formula requires that the function \( f(z) \) being differentiated be analytic in some finite sector of the \( z \)-plane with vertex at \( z = 0 \). We also need \( \oint f(z)dz = 0 \) along any closed path through \( z = 0 \), so that the angle at which \( C \) approaches \( z = 0 \) is arbitrary.

3. Derivatives with respect to any function and partial derivatives

We next assign a meaning to the derivative of order \( \alpha \) with respect to an arbitrary function \( g(z) \) of \( f(z) \), and denote it by the symbol \( D^\alpha_{g(z)} f(z) \). To generate a useful definition, consider the Riemann Liouville integral

\[
D^\alpha_w F(w) = \frac{1}{\Gamma(-\alpha)} \int_0^w \frac{F(\gamma) d\gamma}{(w-\gamma)^{\alpha+1}},
\]
and formally set \( w = g(z) \). We then require \( F(g(z)) = f(z) \)
and \( \mathcal{F} = g(t) \). This is a simple change of variables,
and is all that is needed for our purpose. We obtain

\[
(3.1) \quad D_{g(z)}^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_{g^{-1}(0)}^z \frac{f(t) g'(t) dt}{(g(z) - g(t))^{\alpha+1}},
\]

It is now easy to generalize the Cauchy integral formula
for fractional derivatives (2.2) to include derivatives
with respect to an arbitrary function. We state the
precise form for future reference.

**Definition 3.1**

Let \( f(z) \) be analytic in the simply connected
region \( R \). Let \( g(z) \) be regular and univalent on \( R \), and
let \( g^{-1}(0) \) be an interior or boundary point of \( R \).
Assume also that \( \oint_C f(z) dz = 0 \) for any simple closed
contour \( C \) in \( \mathbb{R} \cup g^{-1}(\infty) \) through \( g^{-1}(0) \). Then if \( \alpha \) is not
a negative integer, and \( z \) is in \( R \), we define the fractional
derivative of order \( \alpha \) of \( f(z) \) with respect to \( g(z) \)
to be

\[
(3.2) \quad D_{g(z)}^\alpha f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{g^{-1}(0)}^{(z^+)} \frac{f(\mathcal{Y}) g'(\mathcal{Y}) d\mathcal{Y}}{(g(\mathcal{Y}) - g(z))^{\alpha+1}}.
\]
For non-integral $\infty$, the integrand has a branch line which begins at $\mathcal{J} = z$ and passes through $\mathcal{J} = g^{-1}(0)$. The notation on this integral implies that the contour of integration starts at $g^{-1}(0)$, encircles $z$ once in the positive sense, and returns to $g^{-1}(0)$ without cutting the branch line or leaving $RU\{g^{-1}(0)\}$.

It is of particular interest to set $g(z) = z-a$. In this case $g^{-1}(0) = a$ and (3.1) becomes

$$D_{z-a}^{\infty} f(z) = \frac{1}{\Gamma(-\infty)} \int_{a}^{z} \frac{f(t) \, dt}{(z-t)^{\infty+1}}.$$  

Other notations have been used by authors of recent papers to denote this last integral, but the above considerations appear to make this notation, presented here for apparently the first time, the most natural.

The reason why the contour in (2.2) defining the generalized Cauchy integral formula, passes through zero is now clear. It must be that

$$\frac{1}{2\pi i} \int_{a}^{(z^+)} \frac{f(\mathcal{J}) \, d\mathcal{J}}{(\mathcal{J} - z)^{\infty+1}} = D_{z-a}^{\infty} f(z).$$
We also require fractional partial derivatives. These have been introduced by M. Riesz [20], and M. A. Bassam [1]. The notation

\[ D_{g(z), h(w)}^{\alpha, \beta} f(z, w) \]

means the fractional derivative of \( f(z, w) \) of order \( \beta \) with respect to \( h(w) \) holding \( z \) fixed, followed by the derivative of order \( \alpha \) with respect to \( g(z) \) holding \( w \) fixed. This is given by

\[ (3.3) \quad D_{g(z), h(w)}^{\alpha, \beta} f(z, w) = \]

\[ \frac{\Gamma(\alpha + 1)}{-4\pi^2} \int_{-\infty}^{(w^+) \Gamma(\beta + 1)} \frac{g'(\xi)}{(g(\xi) - g(z))^{\alpha + 1}} \cdot \frac{f(\xi, \xi)h'(\xi) d\xi d\xi}{(h(\xi) - h(w))^{\beta + 1}} \]

\[ g^{-1}(0) \quad h^{-1}(0) \]

where \( f(z, w) \), \( g(z) \) and \( h(w) \) are assumed (as in the previous definition) to possess sufficient regularity to give the definition meaning.

It is clear from the definition \( (3.3) \) that when

\[ f(z, w) = u(z)v(z), \]

\[ (3.4) \quad D_{g(z), g(w)}^{\alpha, \beta} u(z)v(w) \bigg|_{w=z} = D_{g(z)}^{\alpha} u(z) D_{g(z)}^{\beta} v(z) \]
Thus it is easy to see that (1.2) is a generalization of the Leibniz rule (1.1).

4. The product rule

We next consider extending the elementary Leibniz rule for the derivative of the product of two functions, to fractional derivatives. The formula

\[ D_z^N u(z)v(z) = \sum_{n=0}^{N} \frac{N! D_z^{N-n}u(z) D_z^n v(z)}{(N-n)! \ n!} \]

appears at once to generalize as

\[ D_z^\alpha u(z)v(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1) D_z^{\alpha-n}u(z) D_z^n v(z)}{\Gamma(\alpha-n+1) \ n!} \]

for arbitrary \( \alpha \). This formula can be derived from the Riemann Liouville definition of fractional derivative by integrating by parts. It can also be derived by using the power series method if the functions \( u(z) \) and \( v(z) \) permit.

There is a disturbing feature of (4.1). It is obvious that \( D_z^\alpha uv = D_z^\alpha vu \), but this fact is not clear on the right hand side since \( u \) is differentiated fractionally while \( v \) is differentiated in the usual elementary sense. Could it be that (4.1) is a special case of a more general formula in which the interchangability of the functions \( u \) and \( v \) is obvious? To see that this is the case, formally differentiate
\[ D_{\alpha}^{\gamma} v u = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma+1) D_{\gamma}^{r} v D_{\gamma}^{r} u}{\Gamma(\gamma-r+1) r!} \]

with the operator \( D_{\alpha}^{\gamma} \) and obtain

\[ D_{\alpha}^{\gamma} (D_{\gamma}^{r} v u) = D_{\alpha}^{\gamma} v u = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma+1) D_{\gamma}^{r} (D_{\gamma}^{r-1} u D_{\gamma}^{r} v)}{\Gamma(\gamma-r+1) r!} . \]

Using (4.1) again we obtain

\[ D_{\alpha}^{\gamma} u v = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-r+1) r!} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha-\gamma+1) D_{\gamma}^{r-k} D_{\gamma}^{k} v}{\Gamma(\alpha-\gamma-k+1) k!} \cdot \]

Interchanging the order of summation, and summing diagonally by setting \( n = k-r \) we obtain

\[ D_{\alpha}^{\gamma} u v = \sum_{n=-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{\Gamma(\gamma+1) \Gamma(\alpha-\gamma+1) D_{\gamma}^{r} u D_{\gamma}^{r} v}{\Gamma(\gamma-r+1) \Gamma(\alpha-\gamma-n-r+1)(n+r)! r!} . \]

Using elementary properties of the gamma function and the value of the hypergeometric function of unit argument we can sum over \( r \) to obtain the following generalization of (4.1)
\[
(4.2) \quad \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha + 1) \Gamma(-\gamma - n) \Gamma(\gamma + n)}{\Gamma(\alpha - \gamma - n + 1) \Gamma(\gamma + n + 1)} D_z^\alpha u v
\]

It is now clear that (4.1) is the special case of (4.2) obtained by setting \( \gamma = 0 \). It is also clear that \( u \) and \( v \) can be interchanged in (4.2), and thus the question raised above concerning the existence of a general formula of which (4.1) is a special case has been answered affirmatively. The formula (4.2) can be derived rigorously by expanding \( u \) and \( v \) in power series and following the steps outlined above. However, the proof is lengthy and tedious due to the need to generate inequalities for terms like \( D_z^\alpha f \) to prove the convergence of the series encountered. A much simpler proof is given below using the generalized Cauchy integral formula and contour integration.

After discovering (4.2) the author was informed by a referee that Y. Watanabe [21] had published it in 1931. His proof is in two parts. First he derives (4.1) by expanding \( u(z) \) and \( v(z) \) in power series in \( z \). Since power series by nature converge only in circles, this method demonstrates that (4.1) converges in the largest circle centered at the origin and interior to the full region of convergence [21, p. 12]. He then mistakenly concludes that the series (4.1) converges wherever \( u(z) \)
and \( v(z) \) are analytic, which is not true as is shown below by an example. Finally Watanabe uses (4.1) to derive the general Leibniz rule (4.2) by a method somewhat like that outlined above.

The proof given below using the Cauchy integral formula for fractional derivatives is new. It yields the general Leibniz rule (4.2) as well as the precise region of convergence in one stroke.

**Theorem 4.1 (product rule)**

Let \( u(z) \) and \( v(z) \) be analytic functions of \( z \) on the simply connected region \( R \). Suppose also that 0 is an interior or boundary point of \( R \) and that the integral along any simple closed path in \( R \) through 0 of \( u, v, \) and \( uv \) is zero. Call \( S \) the set of all \( z \) such that the closed disk \( |z| < R \) contains only points \( S \) in \( RU \{ 0 \} \). Then

\[
D_{z}^{\alpha} u(z) v(z) = \sum_{n=-\infty}^{\infty} \binom{\infty}{\gamma+n} D_{z}^{\alpha-\gamma-n} u(z) D_{z}^{\gamma+n} v(z)
\]

for \( z \) in \( S \) and all complex \( \alpha \) and \( \gamma \) for which \( \binom{\infty}{\gamma+n} \) is defined.

**Proof:**
$|\zeta - z| = |z|$
Using the contours shown in Figure 4.1 we know Cauchy's integral formula for fractional
derivatives states that

\[ D_z^\alpha u(z) v(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{C_z} \frac{u(\xi)v(\xi)d\xi}{(\xi - z)^{\alpha + 1}} \]

\[ = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{C_z} \frac{u(\xi)}{(\xi - z)^{\alpha - \gamma + 1}} \frac{v(\xi)}{(\xi - z)^\gamma} d\xi \]

provided \( \alpha \) is not a negative integer. Using the elementary Cauchy integral formula we can reduce this to

\[ D_z^\alpha uv = \frac{\Gamma(\alpha + 1)}{-4\pi^2} \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha - \gamma + 1}} \int_{C_3 - C_1} \frac{v(\eta)d\eta d\xi}{(\eta - z)^\gamma (\xi - \eta)} \]

\[ = \frac{\Gamma(\alpha + 1)}{-4\pi^2} \left\{ \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha - \gamma + 1}} \int_{C_3} \frac{v(\eta)d\eta d\xi}{(\eta - z)^\gamma (\xi - \eta)} + \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha - \gamma + 1}} \int_{C_1} \frac{v(\eta)d\eta d\xi}{(\eta - z)^\gamma (\xi - \eta)} \right\} . \]
In the first term of this last expression, $C_2$ can be replaced by $C_1$ and in the second term, $C_2$ can be replaced by $C_3$. A little straightforward manipulation then yields

$$
D_z \text{uv} = \frac{\Gamma(\alpha+1)}{-4\pi^2} \left\{ \int_{C_1} \frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_3} \frac{v(\zeta)}{(\zeta-z)^{\gamma+1}} \left( 1 - \frac{\xi-z}{\zeta-z} \right) \right. \\
+ \left. \int_{C_3} \frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \frac{v(\zeta)}{(\zeta-z)^{\gamma+1}} \left( 1 - \frac{\xi-z}{\zeta-z} \right) \right\}.
$$

Expanding in power series we have

$$
D_z \text{uv} = \frac{\Gamma(\alpha+1)}{-4\pi^2} \left\{ \int_{C_1} \frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_3} \frac{v(\zeta)}{(\zeta-z)^{\gamma+1}} \left[ \sum_{n=0}^{N} \frac{(\xi-z)^n}{(\zeta-z)} \right] \\
+ \frac{(\xi-z)^{N+1}}{1 - \frac{\xi-z}{\zeta-z}} \right\} d \zeta d \xi \\
\int_{C_3} \frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \frac{v(\zeta)}{(\zeta-z)^{\gamma+1}} \left[ \sum_{n=1}^{N} \frac{(\zeta-z)^n}{(\xi-z)} + \frac{(\zeta-z)^{N+1}}{1 - \frac{\zeta-z}{\xi-z}} \right] d \zeta d \xi \right\}.
$$
\[
\sum_{n=-N}^N \frac{\Gamma (\alpha + 1)}{4\pi^2} \int_{C_2} \frac{u(\xi) \, d\xi}{(\xi - z)^{\alpha - \gamma - n + 1}} \int_{C_2} \frac{v(z) \, d\gamma}{(\xi - z)^{\gamma + n + 1}}
\]

\[
\frac{\Gamma (\alpha + 1)}{4\pi^2} \int_{C_1} \frac{u(\xi)}{(\xi - z)^{\alpha - \gamma}} \int_{C_3} \frac{v(z)}{(\xi - z)^{\gamma + 1}} \frac{\xi - z}{\xi - \xi} \, d\xi \, d\gamma
\]

We see at once from the generalized Cauchy integral formula that \( \sum_{n=-N}^N \) in this last expression is

\[
\sum_{n=-N}^N \frac{\Gamma (\alpha + 1)}{\Gamma (\alpha - \gamma - n + 1) \, \Gamma (\gamma + n + 1)} \frac{D_z}{D_z} u(z) \frac{D_z}{D_z} v(z)
\]

It is an elementary exercise to see that the remaining two terms approach zero as \( N \) grows large. This is because

\[
\left| \frac{\xi - z}{\xi - z} \right| = \left| \frac{\xi - z}{z} \right| < 1
\]
for \( S \) and \( S \) not zero in the first remainder, since \( C_3 \) is a circle. This shows the reason why \( z \in S \). A similar statement holds for the second remainder. Thus the theorem is proved for \( \alpha \) not a negative integer.

If \( \alpha \) is a negative integer, the generalized binomial coefficients \( \binom{\alpha}{\gamma+n} \) are not defined except for integral \( \gamma \). In this case the Cauchy integral formula for fractional derivatives can be replaced in the above equations by its analytic continuation (the Riemann - Liouville integral (2.1)). The proof of the Theorem is now complete.

If the region \( R \) of analyticity of \( u \) and \( v \) is the \( z \)-plane with only isolated points removed, then the region of convergence \( S \) of the generalized Leibniz rule consists of the interior of a polygon whose sides are the perpendicular bisectors of the line segments joining the singularities to the origin. As an example, consider the case where \( \gamma = 0 \), \( u(z) = 1 \), and \( v(z) = \sum_{r=0}^{n} \) \( (a_r - z)^{-1} \). A little computation reveals that

\[
\mathcal{D}_z^{\alpha} \left( \sum_{r=0}^{R} (a_r - z)^{-1} \right) = \frac{\Gamma(\alpha+1)}{\pi \csc \pi \alpha} \sum_{r=0}^{R} \frac{z^{-\alpha}}{a_r - z} \sum_{n=0}^{\infty} \frac{z^n}{(\alpha - n)(z - a_r)^n}.
\]
Figure 4.2
It is easy to see that this series converges for
\[ |z| / |a_r - z| < 1, \ r = 0, 1, 2, \ldots, R. \] This is
the polygon just described. (See Figure 4.2)

In this example (4.2) diverges outside the
closure of the region S described in the Theorem.
Thus with the exception of the boundary of S the precise
region of convergence for general functions \( u(z) \) and \( v(z) \)
has been determined. (Watanabe appears to state that
(4.1) should converge wherever \( u(z) \) and \( v(z) \) are analytic
[21, p. 15, Remark 2]. The above example shows that
this is not correct.)

**Corollary 4.1**

With the hypothesis of the previous theorem
and the additional conditions

(a) \( g(w) \) is analytic for \( w \) in \( g^{-1}(R) \),
(b) \( U(w) = u(g(w)) \),
(c) \( V(w) = v(g(w)) \),

then

\[
(4.3) \quad D^\alpha g(w) U(w) V(w) = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha + 1) D^\alpha g(w) U(w) D^\gamma g(w) V(w)}{\Gamma(\alpha - \gamma n + 1) \Gamma(\gamma + n + 1)}
\]

for \( w \) in \( g^{-1}(S) \).

The proof of this corollary follows immediately
from the previous theorem by replacing \( z \) by \( g(w) \).
As an example, consider the case in which \( \gamma = 0 \), \( u(z) \equiv 1 \), \( v(z) = (1-z)^{-1} \), and \( g(w) = w^2 \). The region \( S \) of convergence of the generalized Leibniz rule (4.3) is \( \text{Re}(z) < \frac{1}{2} \).

In the \( w \) plane we have \( w = \sqrt{z} \) which is the region bound on the right and left by the two branches of the hyperbola \( 2u^2 - 2v^2 = 1 \), where \( w = u + i v \). Therefore (4.3) reduces to

\[
D^\infty \frac{1}{w^2 (1-w^2)} = \frac{1}{\Gamma(-\infty)} \sum_{n=0}^{\infty} \frac{w^{2(n-\infty)}}{(-n) (w^2-1)^{n+1}}
\]

for \( w \) in the region specified.

The region of convergence, \( g^{-1}(S) \), of the Leibniz rule described in the previous corollary will be referred to often in the following pages, and it is therefore convenient to give it the special name "Leibniz region".

**Definition 4.1**

The "Leibniz region of the functions \( U(w) \) and \( V(w) \) with respect to \( g(w) \)" is the region of convergence in the \( w \)-plane of the series (4.3) described in the previous corollary and is denoted by \( L(U,V;g) \).
5. A further generalization of Leibniz rule

To see intuitively how the formula

\[
(5.1) \quad D_z^\infty f(z,z) = \sum_{n=-\infty}^{\infty} \left( \sum_{\nu_n} \infty \right) D_{z,w}^{\infty - \nu_n, \nu_n + \nu} f(z,w) \bigg|_{w=z},
\]

follows from the Leibniz rule

\[
(5.2) \quad D_z^\infty u(z)v(z) = \sum_{n=-\infty}^{\infty} \left( \sum_{\nu_n} \infty \right) D_z^{\infty - \nu_n} u(z) D_z^{\nu_n} v(z),
\]

we expand \( f(z,w) \) in a power series in \( z \) and \( w \)

\[
f(z,w) = \sum a_{r,s} z^r w^s.
\]

Operating with \( D_z^\infty \) and using (5.2) we get

\[
D_z^\infty f(z,z) = \sum a_{r,s} D_z^\infty z^r w^s
\]

\[
= \sum a_{r,s} \sum_{n} \left( \sum_{\nu_n} \infty \right) D_z^{\infty - \nu_n} z^r D_z^{\nu_n} z^s.
\]
Interchanging the order of summation and using (3.4) we get

\[ D_z^\alpha f(z, z) = \sum_n (\gamma + n) \sum_{r, s} a_{r, s} D_z^{\alpha - \gamma - n} z^r D_z^{\gamma + n} z^s \]

\[ = \sum_n (\gamma + n) D_{z, w}^{\alpha - \gamma - n, \gamma + n} \sum_{r, s} a_{r, s} z^r w^s \bigg|_{w=z} \]

This last equation is the desired generalization (5.1).

The steps outlined above could be made into a rigorous proof. However, a far easier proof is obtained by using the Cauchy integral formula for fractional derivatives as demonstrated previously in the proof of the Leibniz rule in Theorem 4.1. In fact, if the reader replaces \( u(\zeta)v(\zeta) \) by \( f(\zeta, \zeta) \) in that proof he will find that (5.1) is obtained without any further modification. We state the precise conclusion as a Theorem.

**Theorem 5.1**

Let \( R \) be a simply connected region in the complex plane. Let the origin be an interior or boundary point of \( R \). Let \( f(z, w) \) be an analytic function for \( z \) and \( w \) in \( R \). Assume also that \( \oint f(z, z)dz, \oint f(z, w)dz, \) and \( \oint f(z, w)dw \) vanish over any simple closed path in \( R \) through the origin. Call \( S \) the set of all \( z \) such that the closed disk
| $\Im - z | \leq | z |$ contains only points $\Im$ in $\mathcal{R} \{0\}$. Then

$$D^\infty_z f(z, z) = \sum_{n=-\infty}^{\infty} \binom{\infty}{\gamma + n} D^\infty_{z, w} f(z, w) \bigg|_{w=z}$$

for $z$ in $\Sigma$ and all complex $\alpha$ and $\gamma$ for which $(\gamma + n)$ is defined.

The more general result in which we differentiate with respect to $g(z)$, (1.2), is obtained immediately from the above theorem by replacing $z$ by $g(z)$ as in Corollary 4.1.

6. Series expansions

We conclude our examination of the generalization of the Leibniz rule by assigning specific values to $f, g, \alpha$, and $\gamma$ in the formula

$$(6.1) \ D^\infty_{g(z)} f(z, z) = \sum_{n=-\infty}^{\infty} \binom{\infty}{\gamma + n} D^\infty_{g(z), g(w)} f(z, w) \bigg|_{w=z}$$

and simplifying the result. Table 6.3 shows the fruit of this procedure. It is not surprising that a wide variety of infinite series expansions is obtained since
the generalized Leibniz rule (6.1) is a glorified integration by parts and many important series are obtained by this technique. For example, the series 2 in Table 6.3 is simply the value of the hypergeometric function $F(a,b;c;z)$ at $z=1$. The series 10 is known as Dougall's formula [2, vol. 1, p.7].

Table 6.2 exhibits the special choices of $f$, $g$, $\alpha$, and $\gamma$ used to generate the corresponding series in Table 6.3. After assigning these choices to (6.1), it is useful to simplify by expressing the resulting fractional derivatives in terms of well known special functions of mathematical physics. For this purpose it is convenient to have a table of special functions expressed as fractional derivatives. A brief list of this type is included in Table 6.1 where the notation $D\alpha_g(z)$ introduced in section 3 is used to advantage. A more extensive table for this purpose is found in [3, vol. 2, pp. 185-212]. The notation for the special functions used is that of Erdelyi at al. [2,3]. Higgins [12] has also compiled an extensive table of special functions represented by fractional derivatives incorporating a different notation.
Listed below are four special forms of the generalized product rule which are obtained from (1.1) by taking specific values for $\alpha$, $\gamma$, and $v$.

These formulas are of interest, for when combined judiciously with a table of fractional integrals or derivatives, a host of series expansions relating special functions is envisioned.

Case 1, $\gamma = 0$.

$$D_\alpha^\gamma u(z)v(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1) u(z)}{\Gamma(\alpha-n+1) n!} \frac{D_n^\gamma v(z)}{D_n^\gamma g(z)}.$$

Case 2, $v \equiv 1$.

$$D_\alpha^\gamma u(z) = \frac{\Gamma(\alpha+1) \sin(\alpha-\gamma)\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n g(z)^n \Gamma(\gamma+n+1)}{(\alpha-\gamma-n) \Gamma\left(\frac{\gamma+n}{\gamma+n+1}\right)} u(z).$$

Case 3, $\alpha = 0$.

$$u(z)v(z) = \frac{\sin \gamma \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n D_n^\gamma u(z)}{D_n^\gamma v(z)} \frac{D_n^\gamma v(z)}{D_n^\gamma g(z)}.$$

Case 4, $\alpha = 0$, $v \equiv 1$.

$$u(z) = \frac{\sin^2 \gamma \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{\Gamma(\gamma+n) g(z)}{\gamma + n} D_n^\gamma u(z).$$
<table>
<thead>
<tr>
<th>NAME</th>
<th>DERIVATIVE REPRESENTATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypergeometric function</td>
<td>$F(a,b;c;z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(b) z} \frac{D^{b-c}}{z} \frac{z^{b-1}}{(1-z)^a}$</td>
</tr>
<tr>
<td>Confluent hypergeometric function</td>
<td>$,^{1}_1F_1(a;c;z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(a)} D^{a-c}z^{a-1} z$</td>
</tr>
<tr>
<td>Bessel function</td>
<td>$J_v(z) = \pi^{-1/2} (2z)^{-v} D\frac{-v-1/2}{z} \cos z/z$</td>
</tr>
<tr>
<td>Modified Bessel function</td>
<td>$I_v(z) = \pi^{-1/2} (2z)^{-v} D\frac{-v-1/2}{z} \cosh z/z$</td>
</tr>
<tr>
<td>Struve function</td>
<td>$H_v(z) = \pi^{-1/2} (2z)^{-v} D\frac{-v-1/2}{z} \sin z/z$</td>
</tr>
<tr>
<td>Modified Struve function</td>
<td>$L_v(z) = \pi^{-1/2} (2z)^{-v} D\frac{-v-1/2}{z} \sinh z/z$</td>
</tr>
<tr>
<td>Legendre function of the first kind</td>
<td>$P_v(z) = D\frac{v}{1-z} (1-z^2)^v / (\Gamma(v+1) 2^v)$</td>
</tr>
<tr>
<td>Associated Legendre function of the first kind</td>
<td>$P^{u}_v(z) = \frac{(1-z^2)^{u/2}}{\Gamma(v+1) 2^v} D\frac{v+u}{1-z} (1-z^2)^v$</td>
</tr>
<tr>
<td>Laguerre function</td>
<td>$L_v(a)(z) = \frac{\Gamma(a+v+1)z^{-a}}{\Gamma(v+1) \Gamma(-v)} D\frac{-a-v-1}{z} e^{z} z^{-v-1}$</td>
</tr>
<tr>
<td>Incomplete Gamma function</td>
<td>$\gamma(a,z) = \Gamma(a) e^{-z} D\frac{-a}{z} e^{z}$</td>
</tr>
<tr>
<td>Series Number</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>---------------</td>
<td>--------------</td>
</tr>
<tr>
<td>1</td>
<td>b-c</td>
</tr>
<tr>
<td>2</td>
<td>-a</td>
</tr>
<tr>
<td>3</td>
<td>a-c</td>
</tr>
<tr>
<td>4</td>
<td>-v-1/2</td>
</tr>
<tr>
<td>5</td>
<td>-v-1/2</td>
</tr>
<tr>
<td>6</td>
<td>-v-1/2</td>
</tr>
<tr>
<td>7</td>
<td>-v-1/2</td>
</tr>
<tr>
<td>8</td>
<td>v</td>
</tr>
<tr>
<td>9</td>
<td>b+B-d-D</td>
</tr>
<tr>
<td>10</td>
<td>c-a-l</td>
</tr>
<tr>
<td>11</td>
<td>a-b</td>
</tr>
<tr>
<td>12</td>
<td>-v-1/2</td>
</tr>
<tr>
<td>13</td>
<td>-v-1/2</td>
</tr>
<tr>
<td>Series Number</td>
<td>$\alpha$</td>
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<tr>
<td>---------------</td>
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</tr>
<tr>
<td>14</td>
<td>$-v-1/2$</td>
</tr>
<tr>
<td>15</td>
<td>$-v-1/2$</td>
</tr>
<tr>
<td>16</td>
<td>$v+u$</td>
</tr>
<tr>
<td>17</td>
<td>$-a-v-1$</td>
</tr>
<tr>
<td>18</td>
<td>$-a$</td>
</tr>
<tr>
<td>19</td>
<td>$-a$</td>
</tr>
<tr>
<td>20</td>
<td>$-a$</td>
</tr>
<tr>
<td>21</td>
<td>$c-a-1$</td>
</tr>
</tbody>
</table>
TABLE 6.3
SERIES EXPANSIONS OF SPECIAL FUNCTIONS DERIVED FROM
THE GENERALIZATION OF THE LEIBNIZ RULE

1. \[ F(a, b; c; z) = \frac{\Gamma(c) \Gamma(b-c+1)\sin\pi(b-c)}{\pi} \]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n F(a, b; b-n; z)}{\Gamma(b-n) n! (b-c-n)} , \quad \text{Re}(b)>0, \text{Re}(z)<1/2. \]

2. \[ \frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} , \quad \text{Re}(c-a-b)>0. \]

3. \[ _1F_1(a; c; z) = \frac{\Gamma(c) \Gamma(a-c+1) \sin(a-c)\pi}{\pi} \]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n _1F_1(a; a-n; z)}{\Gamma(a-n) n! (a-c-n)} , \quad \text{Re}(a)>0. \]

4. \[ F_v(z) = \frac{\Gamma(1/2-v) \cos\pi v}{\pi 2^{v+1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+v+1/2} F_{-v+n+1/2}(z)}{2^n (v+n+1/2) n!} \]
where \( F_v = J_v, I_v, H_v, \) and \( L_v, \) respectively
for series 4, 5, 6, and 7.

8. \[ P_v(z) = \frac{\sin\pi v}{\pi 2^v} \sum_{n=0}^{\infty} \frac{(-2)^n (1-z)^{n-v} P_n(z)}{v-n} , \quad \text{Re}(v)>-1, \text{Re}(z)>0. \]
TABLE 6.3

(Continued)

9. \[ F(a+A, b+B-1; d+D-1; z) = \frac{\Gamma(b+B-d-D+1) \Gamma(d+D-1) \Gamma(b) \Gamma(B)}{\Gamma(b+B-1)} \] \[ \sum_{n=-\infty}^{\infty} \frac{F(a,b;d+n;z)}{\Gamma(b-d-n+1) \Gamma(B-D+n+1) \Gamma(d+n) \Gamma(D-n)} \] \[ \text{Re}(b+B)>1, \text{Re}(b)>0, \text{Re}(B)>0, \text{Re}(z)<1/2. \]

10. \[ \csc^\alpha \csc^\beta \frac{\pi^2}{\Gamma(d-b)} \Gamma(c+d-a-b-1) = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n) \Gamma(d+n)} \] \[ \text{Re}(c+d-a-b)>1. \]

11. \[ \frac{\Gamma(a-b+1) \Gamma(b) \sin(a-b-c)\pi}{\pi} \] \[ \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(a-b-c-n) \Gamma(c+n+1) \Gamma(a-c-n)} \] \[ \text{Re}(a)>0. \]

12. \[ \tilde{\mathcal{F}}_v(z) = \frac{\Gamma(\frac{1}{2} - v) \sin(b-v)\pi}{\pi (2/z)^{v-b}} \sum_{n=-\infty}^{\infty} \frac{(-z)^n}{2^n(b-v-n) \Gamma(\frac{1}{2} - b+n)} \] to \[ \tilde{\mathcal{H}}_v(z) \] 15. where \( \tilde{\mathcal{F}}_v(z) = \tilde{J}_v(z), \tilde{I}_v(z), \tilde{H}_v(z), \text{and} \tilde{L}_v(z) \) for 12, 13, 14, and 15 respectively.
TABLE 6.3
(Continued)

16. \( p_v^u(z) = \frac{\Gamma(v+u+1) \sin(v+u-c)\pi}{\pi} \times \)

\[\sum_{n=-\infty}^{\infty} \frac{(-1)^n p_v^{c-v+n}(z)}{n+c+1}(v+u-c-n) (\frac{l-z}{l+z})^{(n+c-u-v)/2}, \]

\( \text{Re}(v) > -1, \text{Re}(z) > 0. \)

17. \( L_v^{(a)}(z) = -\frac{\sin(a+v+c)\pi}{\sin(a+v)\pi} \sum_{n=-\infty}^{\infty} \frac{(-n-v-c)}{n+a+v+c} L_v^{(a)}(z), \)

\( \text{Re}(v) < 0. \)

18. \( \mathcal{J}(a,z) = -\frac{\sin(a+c)\pi}{\sin(a\pi)} \sum_{n=-\infty}^{\infty} \frac{z^{n+c+a}}{a+c+n} \mathcal{J}(-c-n,z) \).

19. \( \frac{\Gamma(c) \Gamma(c-a-b) (1-x)^{-b}}{\Gamma(c-a) \Gamma(c-b)} = \)

\[\sum_{n=0}^{\infty} \frac{(a)_n (b)_n F(b+n,c-a;c+n;x)}{(c)_n n!}, \]

\( x \neq -1, \text{Re}(c-a-b) > 0, \text{Re}(c-a) > 0, \text{Re}(1-b) > 0. \)
TABLE 6.3

(Continued)

\[
\frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \binom{k+1}{a+b+k+1} \binom{b}{k+1}, \binom{b+1}{k+1}, \ldots, \binom{b+k}{k+1} ; z^{k+1} \\
= \sum_{n=0}^{\infty} \frac{\Gamma(b+nk) \Gamma(\alpha+n)(-z^{k+1})^n}{\Gamma(a+b+kn+n) \Gamma(a+b+kn+n+n)}
\]

\[
\binom{\frac{b+nk}{k}, \frac{b+nk+1}{k}, \ldots, \frac{b+nk+k-1}{k}}{k+1} ; z^{k+1} \\
= \frac{\prod_{\text{Re}(b)>0}}{\Gamma(c-d-a-b-1) \csc \pi a \csc \pi b} \\
\binom{e, \frac{c+d-a-b-1}{2}, \frac{c+d-a-b}{2}, \frac{d-b}{2}, \frac{d-b+1}{2}, -z^2} \\
= \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n) \Gamma(d+n)} \binom{e, d-a, c-b, d+n, 1-b-n}{-z^2} \\
\text{Re}(d-a-e)>0, \text{Re}(c-b)>0, \text{Re}(1-e)>0, \text{Re}(c+d-a-b-1)>0, \\
|\text{Im}(z)| < 1/2.
CHAPTER II

THE FRACTIONAL DERIVATIVE OF A COMPOSITE FUNCTION

7. Introduction

In the elementary calculus one considers the derivative of order \( N \) of the composite function \( f(z) = F(h(z)) \) and obtains the formula \([7, \text{ p. 19}]\)

\[
(7.1) \quad D_z^N f(z) = \sum_{n=0}^{N} U_n(z) D_h(z)^n f(z) /n!,
\]

where

\[
U_n(z) = \sum_{r=0}^{n} \binom{n}{r} (-h(z))^r D_z^N h(z)^{n-r}.
\]

In this chapter we consider the extension of \((7.1)\) to fractional derivatives. We derive the fundamental result
\( (7.2) \quad D^\alpha_{g(z)} f(z) = \left. D_h(z) \left\{ \frac{f(z)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha + 1} \right\} \right|_{w=z} , \)

where the notation \( D^\alpha_{g(z)} f(z) \) means the fractional derivative of order \( \alpha \) of \( f(z) \) with respect to \( g(z) \).

The Leibniz rule for fractional derivatives is then applied to \((7.2)\) to obtain the new series expansion

\( (7.3) \quad D^\alpha_{g(z)} f(z) = \sum_{n=-\infty}^{\infty} \left( \gamma + n \right)_n \frac{f(z)}{F(z,w)} \times D^{\alpha - \gamma - n}_{h(z)} \left\{ \frac{F(z,w)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha + 1} \right\} \bigg|_{w=z} , \)

where

\[
\left( \gamma + n \right)_n = \frac{\Gamma(\gamma + 1) \Gamma(\gamma + n + 1)}{\Gamma(\gamma + n + 1 + 1) \Gamma(\gamma + 1 + n + 1)} .
\]

The formula \((7.1)\) from the elementary calculus is shown
to be a special case of the generalized chain rule (7.3)

A few specific examples of (7.3) are examined.

Novel derivations of the known result

\[ F(\alpha, 1 - \alpha; p - \alpha + 1; \frac{1}{2}) = \frac{2^\alpha \Gamma(p - \alpha + 1) \Gamma(p/2 + 1)}{\Gamma(p/2 - \alpha + 1) \Gamma(p + 1)} , \]

and Kummer's formula

\[ F(\alpha + 1, p; p - \alpha; -1) = \frac{\Gamma(p/2) \Gamma(p - \alpha)}{2 \Gamma(p) \Gamma(p/2 - \alpha)} \]

are obtained as well as new results.
8. The generalized chain rule

We begin by deriving the fundamental relation

\[
\frac{\partial^n}{\partial y^n} \frac{\partial^{n-1}}{\partial x^{n-1}} f(x, y) = \frac{\partial^n}{\partial y^n} \int_0^y \frac{\partial^{n-1}}{\partial x^{n-1}} f(x, y) \, dx
\]

This relation combined with the Leibniz rule will yield the generalized chain rule for fractional derivatives.

**Theorem 8.1**

Let \( f(g^{-1}(z)) \) and \( f(h^{-1}(z)) \) be defined and analytic on the simply connected region \( R \), and let the origin be an interior or boundary point of \( R \). Suppose also that \( g^{-1}(z) \) and \( h^{-1}(z) \) are regular univalent functions on \( R \), and that \( h^{-1}(0) = g^{-1}(0) \). Let \( \oint g^{-1}(z) \, dz \) vanish over every simple closed contour in \( R \) through the origin. Then the fundamental relation (8.1) is valid.
Proof

The result follows immediately upon converting both sides of the fundamental relation (8.1) to contour integrals by means of the definition of fractional derivative (3.2).

The Leibniz rule can be applied to the right hand side of (8.1) once we select \( u(z) \) and \( v(z) \) such that

\[
u(z)v(z) = \frac{f(z)g'^{(z)}}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1}.
\]

To indicate all possible methods of factoring, we introduce the arbitrary function \( F(z,w) \) and set

\[
(8.2) \quad u(z) = \frac{F(z,w)g'(w)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1}
\]

and

\[
v(z) = \frac{f(z)}{F(z,w)}.
\]

The Leibniz rule (1.1) yields the desired generalized chain rule at once.
\[(8.3) \quad D_{g(z)} f(z) = \sum_{n=-\infty}^{\infty} \left( \frac{D_{h(z)}^{\gamma+n}( f(z) )}{F(z,w)} \right) \times \]
\[
\left( \frac{F(z,w)g'(z)}{h'(z)} \right) \left( \frac{h(z)-h(w)}{g(z)-g(w)} \right)^{n+1} \bigg|_{w=z} \]

The precise conclusion is stated as a Theorem.

**Theorem 8.2**

Let \( f(z), g(z) \) and \( h(z) \) satisfy the conditions of Theorem 8.1. Let \( F(h^{-1}(z), h^{-1}(w)) \) be regular on \( R \times R \). Let \( u(z) \) and \( v(z) \) be defined by (8.2) and satisfy the conditions for the application of the Leibniz rule to \( D_{h}^{\alpha} uv \) as stated in Corollary 4.1. Then the generalized chain rule (8.3) is valid for \( z \) in the Leibniz region \( L(u,v;h) \) and arbitrary \( \gamma \).

We conclude the analytical investigation of the generalized chain rule by converting (8.3) into a form somewhat like the elementary calculus formula (7.1). The new form is

\[(8.4) \quad D_{g(z)}^{\infty} f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} D_{g(z)}^{\infty} \left( \frac{F(z,w)(h(z)-h(w))}{g(z)-g(w)} \right)^{n} \times \]
\[
\left. D_{h(z)}^{n} \frac{f(z)}{F(z,w)} \right|_{w=z} \]
where

\[(8.5) \quad \left. \frac{D^\infty_{g(z)} P(z,w)(h(z)-h(w))}{w=z} \right|_{w=z} = \sum_{r=0}^{n} \binom{n}{r} (-1)^r h(z)^r \left. \frac{D_{g(z)}^\infty P(z,w)}{w=z} \right|_{w=z}^{h(z)-h(w)^{n-r}}.\]

The elementary calculus formula (7.1) is seen as the special case in which \( \infty \) is an integer, \( g(z) = z \), and \( F(z,w) = 1 \).

**Theorem 8.3**

With the hypothesis of Theorem 8.2, the relations (8.4) and (8.5) are valid.

**Proof**

Set \( \gamma = 0 \) in the generalized chain rule (8.3). The summation now extends from \( n = 0 \) to \( \infty \) rather than from \( -\infty \) to \( \infty \). We see at once that

\[
\left( \binom{\infty}{n} \right) \frac{D_{h(z)}^\infty -n}{D_{h(z)}^\infty} \left. \left( \frac{P(z,w)g^i(z)}{h^i(z)} \left( \frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\infty+1} \right) \right|_{w=z}^{h(z)-h(w)}
\]

\[
= \frac{1}{n!} \left. D_{g(z)}^\infty P(z,w)(h(z)-h(w))^n \right|_{w=z}
\]
upon writing both sides as a contour integral by (3.2). The relation (8.5) is obtained at once upon expanding \((h(z)-h(w))^n\) by the binomial theorem.

It is useful to note that Theorem 8.3 is valid even when \(h^{-1}(0)\) and \(g^{-1}(0)\) are not equal. This is seen at once upon replacing \(h(z)\) by \(h(z)-h(g^{-1}(0))\) in (8.4) and observing that \(h'(z)\) and \(D_h^n(z)\) do not change.

We have demonstrated that the formulas from the elementary calculus for the derivatives of a composite function generalize to fractional derivatives in a natural way. We proceed to investigate some consequences of the generalized chain rule through the study of a few specific examples.

4. Examples

We conclude this chapter with an examination of a few special cases of the generalized chain rule. These require the evaluation of the fractional derivatives of elementary functions. A table of fractional derivatives or integrals such as that found in [3, vol.2, pp. 185-214] is useful for this purpose.

The notations for the special functions used is that of Erdelyi et al. [2, 3].

Example 1.

Setting \(f(z) = z^{p-2}\), \(g(z) = z^2\) and \(h(z) = z\) in the fundamental relation (8.1) we obtain
\[
D \left( z^p \right)_{z^2} \bigg|_{w=z} = D_z 2z^{p-1}(z+w)^{-\alpha-1} \bigg|_{w=z} = z^{p-2}
\]

The left hand side is evaluated with the aid of the relation

\[
D_z z^q = \frac{\Gamma(q+1) z^{q-p-\alpha}}{\Gamma(q-\alpha+1)},
\]

after replacing \(z\) by \(z^2\). Using [3, vol. 2, p. 186, no. 9] we get Kummer's formula

\[
\mathcal{F}(\alpha+1, p; p-\alpha; -1) = \frac{\Gamma(p/2) \Gamma(p-\alpha)}{2 \Gamma(p) \Gamma(p/2 - \alpha)}.
\]

Example 9.2

Letting \(f(z) = z^p\), \(g(z) = z^2\), \(h(z) = z\), \(F(z,w) = z^p(z+w)^{\alpha+1}\) and \(\gamma = 0\) in the generalized chain rule (8.3) we obtain

\[
\frac{\Gamma(p/2 + 1)}{\Gamma(p/2 - \alpha + 1)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha+n) \Gamma(p+1)(-)^n(2)^{-\alpha-n}}{\Gamma(1-\alpha-n) \Gamma(p-\alpha+n+1) n!},
\]

which quickly reduces to

\[
(9.1) \quad \frac{\Gamma(p-\alpha+1) \Gamma(p/2+1) z^{\alpha}}{\Gamma(p/2-\alpha+1) \Gamma(p+1)} = \mathcal{F}(\alpha, 1-\alpha; p-\alpha+1; \frac{1}{2}).
\]
Note that this novel method for determining the known relation (9.1) does not require the integral representation for the hypergeometric function. It provides a direct evaluation of the hypergeometric series of argument $z = \frac{1}{k}$.

Example 9.3

Setting $g(z) = z$, $h(z) = z^k$ and $F(z, w) = z^q$ in (3.4) we obtain

$$D_z^\infty f(z) = \frac{\Gamma(q+1)z^{q-\infty}}{\Gamma(q-\infty +1)} \sum_{n=0}^{\infty} \frac{(-z^k)^n}{n!} D_{z^k}^n f(z)z^{-q}$$

$$\times \sum_{k+1}^q \sum_{\frac{q+1}{k}, \frac{q+2}{k}, ..., \frac{q+k}{k}} \sum_{\frac{q-\infty+1}{k}, \frac{q-\infty+2}{k}, ..., \frac{q-\infty+k}{k}} 1$$

with the aid of [3, vol.2, p. 186, no. 11]. This is the generalized chain rule for the fractional derivative of the composite function $f(z) = F(z^k)$ in terms of derivatives with respect to $z^k$. 
Example 9.4

Setting \( g(z) = z^p \), \( h(z) = z \), and 
\( F(z,w) = z^{q-p+1} \) in (8.4) we obtain

\[
\frac{D^\infty_{z^p}}{z^p} f(z) = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{D^\infty_{z^p} z^q(w-z)^n}{w=z} \times D^n_z f(z)z^{p-q-1}.
\]

Using the Cauchy integral formula for fractional derivatives (3.2) it is easy to see that

\[
\frac{(-)^n}{n!} \frac{D^\infty_{z^p} z^q(w-z)^n}{w=z} =
\]

\[
\frac{(-)^{n-\alpha-1}}{\Gamma(-\alpha)} \frac{D^{n-1}_{z^p} (z^p-w^p)^{-\alpha-1} z^q}{w=z}.
\]

Finally, using [3, vol.2, p. 186, no. 11] we get
\[ (9.2) \quad \frac{D_{z^p}^\alpha f(z)}{p} = \frac{p}{\Gamma(q+1)} \frac{z^{q-\alpha} p-p+1}{\Gamma(-\alpha)} \times \]

\[ \sum_{n=0}^{\infty} \frac{(-z)^n D_z f(z) z^{p-q-1}}{\Gamma(q+n+2)} \frac{\frac{\Gamma(p+1)}{p} (\alpha+1, \frac{q+1}{p}, \frac{q+2}{p}, \ldots, \frac{q+p}{p})}{\frac{q+n+2}{p}, \frac{q+n+3}{p}, \ldots, \frac{q+n+p+1}{p}; 1} \]

Computation of the coefficient of \( D_z^n f(z) z^{p-q-1} \) by means of equation (9.5) rather than by the procedure outlined above reveals that

\[ p+1 \frac{\Gamma(p+1)}{p} (\alpha+1, \frac{q+1}{p}, \frac{q+2}{p}, \ldots, \frac{q+p}{p}) \]

\[ \frac{q+n+2}{p}, \frac{q+n+3}{p}, \ldots, \frac{q+n+p+1}{p}; 1. \]

equals the finite sum of gamma functions

\[ \frac{\Gamma(-\alpha)}{\Gamma(q+1)} \frac{\Gamma(q+n+2)}{\Gamma(q+1)} \sum_{r=0}^{n} (-)^{n+r} \frac{\Gamma(q+n-r+1)/p}{(n-r)! \Gamma(q+n+1-p\alpha)/p} \]

Replacing \( z \) by \( z^{1/p} \) in equation (9.2) yields the generalized chain rule for the fractional derivative of
f(z^{1/p}) in terms of derivatives with respect to z^{1/p}.
CHAPTER III
THE TAYLOR - RIEMANN THEOREM AND THE DISCOVERY OF
GENERATING FUNCTIONS

10. Introduction

In this chapter, the region in the z-plane in which the Taylor - Riemann formula

\[ f(z) = \sum_{n=-\infty}^{\infty} \frac{\gamma+n}{\Gamma(\gamma+n+1)} \frac{D_{z-b}^\alpha f(z)|_{z=a}}{(z-a)^{\gamma+n}} \]

(10.1)

converges pointwise to the function \( f(z) \) is derived for the first time. The notation \( D_{z-b}^\alpha f(z) \) denotes the fractional derivative of order \( \alpha \) of \( f(z) \) with respect to \( z-b \), and \( \gamma, \alpha, b, \) and \( z \) are arbitrary complex numbers. The Taylor - Riemann formula reduces to the elementary Taylor’s expansion of \( f(z) \) about the point \( z=a \) when \( \gamma=0 \). The Taylor - Riemann theorem is then generalized and used to discover certain generating functions for the special functions of mathematical physics. In particular, it is shown that any special function \( \mathcal{F}(\alpha; z) \) represented by a fractional derivative in the form
\( \mathcal{A}(\alpha; z) = K(\alpha)H(z)G(z)D^{p+b}\alpha_{g(z)f(z)}, \)

(where \( H, G, f \) and \( g \) are elementary functions independent of \( \alpha \),) has a simple generating function which can be written explicitly as

\[
(10.3) \quad t^{-\frac{\gamma}{b}} H(w) G(w)^{-p/b} f(\left(g^{-1}(g(w) + tG(w)^{1/b})\right) = \\
\sum_{n=-\infty}^{\infty} \frac{\mathcal{A}\left(\frac{\gamma-p+n}{b}; w\right) t^n}{K\left(\frac{\gamma-p+n}{b}\right) \Gamma(\gamma+n+1)}.
\]

A generalization of (10.3) is also given which not only produces many familiar generating functions, but also demonstrates a connection between hitherto seemingly unrelated results. New generating functions are also obtained.

The generalized Taylor theorem (10.1) was written by Riemann [19] in 1876 in an attempt to invent a definition of generalized derivative. As a special case of (10.1) Riemann found the generalized binomial theorem.
\[(10.4) \quad (a+b)^\alpha = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha+1) a^{\alpha-n} b^{\gamma+n}}{\Gamma(\alpha-\gamma-n+1) \Gamma(\gamma+n+1)}\]

which assumes the familiar form when \( \gamma = 0 \). This formula (10.4) appeared in 1899 in the work of Heaviside [10], and in 1931 in the work of Watanabe [21, 22]. The first critical examination of the Taylor-Riemann formula (10.1) itself, did not occur until 1945 when G. H. Hardy [9] discussed the series (10.1) as an asymptotic expansion of \( f(z) \), and as a series summable (Borel) to \( f(z) \). In Hardy's work, \( a, \gamma \), and \( z \) are real, and \( b \) assumes the two specific values 0 and \(-\infty\). The question of the pointwise convergence of the Taylor - Riemann formula to the function \( f(z) \) in the complex \( z \)-plane is considered in this paper for the first time.

In this chapter the Taylor - Riemann formula is investigated with the aid of a Cauchy integral formula for fractional derivatives. Although Watanabe [22] used contour integrals for the study of Riemann's binomial theorem, this method is applied to the general theorem (10.1) for apparently the first time here.

It is hoped that the simple way in which the fractional derivative representation yields a genera-
ing function through the Riemann - Taylor formula will demonstrate some of the advantages of using fractional derivatives.

11. The Taylor - Riemann formula

The Cauchy integral formula for fractional derivatives greatly simplifies the examination of the following three theorems on the Taylor-Riemann formula.

Theorem 11.1

Let \( f(z) = (z-b)^\infty h(z) \), where \( h(z) \) is analytic for \( |z-a| < r \). Then

\[
(11.1) \quad f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{\infty+n} f(z)}{\Gamma(\infty+n+1)} \bigg|_{z=a} (z-a)^{\infty+n}
\]

for \( \text{Re} (\infty) > -1 \), and \( z \) in the annulus \( A \) defined by \( |b-a| < |z-a| < r \).
Proof:

Since

\[ f(z) (z-a)^{-\infty} = h(z) \left( \frac{z-b}{z-a} \right)^{\infty} \]

is analytic in the annulus \( A \), it can be expanded there in a Laurent series in powers of \( z-a \).

\[
  f(z) (z-a)^{-\infty} = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} \int_{a^+}^{(t-a)^{\infty} + n+1} \frac{f(t)dt}{(t-a)^{\infty} + n+1} (z-a)^n .
\]

The comparison of the coefficient of \( (z-a)^n \) with (3.2) yields the formula (11.1) at once.

**Theorem 4.2**

With the hypothesis of Theorem 1 and \( \text{Re}(\alpha) > -1 \),
\[(11.2) \quad f(z) = \sum_{n=-\infty}^{\infty} \frac{\frac{\gamma+n}{\beta-b} f(z)}{(z-a)^{\gamma+n}} \frac{(\gamma+n+1)}{\gamma+n+1} \]

where \(\gamma\) is arbitrary and \(z\) is on the circle \(|z-a| = |b-a|\), \(z \neq b\).

**Proof:**

Proceeding as in the proof of the previous Theorem, we expand

\[f(z) (z-a)^{-\gamma} = \left(\frac{z-b}{z-a}\right)^{\gamma} (z-b)^{\alpha-\gamma} h(z)\]

in a Laurent series in powers of \(z-a\). The factor \((z-a)^{-\gamma} (z-b)^{\gamma}\) has a branch line from \(a\) to \(b\), while the factor \((z-b)^{\alpha-\gamma}\) has a branch line from \(b\) to infinity. Therefore the annulus of convergence for our Laurent series has width zero. (The series actually converges for \(z\) on the circle \(|z-a| = |b-a|\), \(z \neq b\), by Fourier's theorem [244, p. 175], and the substitution \(|z-a| = |b-a| \exp(i\Theta)\).)
Theorem 11.3

With the hypothesis of Theorem 11.1 and \( \Re(\alpha - \beta) > -1 \),

\[
(11.3) \quad D_{z-b}^\alpha f(z) = \sum_{n=-\infty}^{\infty} D_{z-b}^{\gamma+n} f(z) \bigg|_{z=a} (z-a)^{\gamma-\beta+n} T(\gamma)^n \gamma+n+1
\]

where \( z \) is on the circle \(|z-a| = |b-a|, z \neq b\), and \( \gamma \) is arbitrary. If \( \gamma = \infty \), then the series converges in the annulus \(|b-a| < |z-a| < r\).

Proof:

\[
D_{z-b}^\alpha f(z) = (z-b)^{\alpha-\beta} h^*(z)
\]

where \( h^*(z) \) is analytic for \(|z-a| < r\). Replace \( f(z) \) by \( D_{z-b}^\alpha f(z) \) in the two previous Theorems. Using the fact that

\[
D_{z-b}^{\gamma+n} D_{z-b}^\alpha f(z) = D_{z-b}^{\gamma+\beta+n} f(z),
\]
(see [4, p. 18]): the result (11.3) follows after replacing \( \gamma \) by \( \gamma - \beta \).

Having developed the analytical machinery for the generalized Taylor - Riemann formula (11.3), we proceed to apply it to the discovery of generating functions for the special functions of mathematical physics.

12. The discovery of generating functions.

The discovery of generating functions has been, for the most part, a matter of skillful manipulations. While the methods presented below for applying the Taylor - Riemann theorem with the fractional derivative representations for special functions certainly does not produce all generating functions, it does give several of the most important ones and some that appear to be new. Furthermore, our technique reveals a connection between hitherto seemingly unrelated results. This will be demonstrated by specific examples in the next section.

Suppose \( F(\alpha ; z) \) is a special function of the variable \( z \) and parameter \( \alpha \) which can be represented by means of fractional differentiation by
(12.1) \[ \mathcal{I}(\alpha ; z) = \mathcal{K}(\alpha) H(z) G(z)^{\alpha} \frac{D^{p+b}\alpha}{g(z)} f(z). \]

Here \( g, f, H, \) and \( G \) are elementary functions independent of \( \alpha \), while \( p \) and \( b \) are constants. (A brief list of such representations is given in Table 19.1.) Rewriting formally the generalized Taylor–Riemann formula (11.3) in a slightly different notation we have

\[ (12.2) \quad \frac{D^{\beta}}{x} f(g^{-1}(x)) = \sum_{n=-\infty}^{\infty} \frac{D_{x} f(g^{-1}(x))}{(x-a)^{\gamma-n}} \bigg|_{x=a} \frac{(x-a)^{\gamma-\beta+n+1}}{\Gamma(\gamma-\beta+n+1)}. \]

Replacing \( g(z) \) by \( x \) in (12.1) we have formally

\[ (12.3) \quad \mathcal{I}(\alpha, g^{-1}(x)) = \mathcal{K}(\alpha) H(g^{-1}(x)) G(g^{-1}(x))^{\alpha} \frac{D^{p+b}\alpha}{x} f(g^{-1}(x)). \]

A simple substitution of (12.3) into (12.2) followed by a brief manipulation and the change of variables

\[ t = (x-a) G(g^{-1}(a))^{-1/b} \quad \text{and} \quad w = g^{-1}(a), \]

yields formally
\[
\begin{align*}
\frac{t^{-\gamma} H(w) G(w)^{(\beta-p)/b} \mathcal{F}(((\beta-p)/b; g^{-1}(g(w)+tG(w)^{1/b}))}{K((\beta-p)/b)H(g^{-1}(g(w)+tG(w)^{1/b}))G(g^{-1}(g(w)+tG(w)^{1/b}))^{(\beta-p)/b}} \\
= \sum_{n=-\infty}^{\infty} \frac{\mathcal{F}(((\gamma-p+n)/b; w) t^n}{K((\gamma-p+n)/b) \Gamma(\gamma-\beta+n+1)} .
\end{align*}
\]

The details of this computation involve only elementary algebra and are mercifully omitted.

(12.4) is a generating function for the functions \( \mathcal{F}((\gamma-p+n)/b; w) \), where \( n \) is any integer, in terms of \( \mathcal{F}((\beta-p)/b; -) \). Since \( D_x^0 f(x) = f(x) \), (12.2) reveals that the special case of (12.4) in which \( \beta = 0 \) is

\[
\begin{align*}
\frac{t^{-\gamma} H(w) G(w)^{-p/b} f(g^{-1}(g(w)+tG(w)^{1/b}))}{K((\gamma-p+n)/b) \Gamma(\gamma+n+1)} \\
= \sum_{n=-\infty}^{\infty} \frac{\mathcal{F}(((\gamma-p+n)/b; w) t^n}{K((\gamma-p+n)/b) \Gamma(\gamma+n+1)} .
\end{align*}
\]

This is a generating function for \( \mathcal{F} \) in terms of elementary functions. Note also that the special cases \( \gamma = \beta \) in (12.4) and \( \gamma = 0 \) in (12.5) cause the summation to go from
\[ n = 0 \text{ to } \infty \text{ rather than from } -\infty \text{ to } \infty . \]

The formulas (12.4) and (12.5) permit us to write many generating functions at once from the fractional derivative representation (12.1). Besides the brief list of such representations in Table 13.1, the reader can construct many more from the table of fractional derivatives in [3, vol. 2, pp. 185-214]. We conclude the presentation by giving explicit examples of generating functions obtained formally from (12.4) and (12.5).

13. Examples

The fractional derivative representations for special functions in Table 12.1 will now be used in conjunction with the formulas (12.4) and (12.5) to determine explicit generating functions. The notations for the special functions used here are those of Erdelyi et. al. [2,3].
Example 13.1

The representation for the Bessel function $J_\infty(z)$ from Table 13.1, in conjunction with formula (4.2.4), yields

\[
\left( \frac{w+2t}{w} \right) \frac{\beta}{2} - \frac{1}{4} J_{-\beta - \frac{1}{2}} \left( \sqrt{w^2 + 2wt} \right) =
\]

\[
\sum_{n=\infty}^{\infty} \frac{J_{\gamma-n-\frac{1}{2}}(w) t^n}{\Gamma(\gamma-\beta+n+1)}.
\]

(13.1)

The special case of (13.1) in which $\gamma = \beta$ is known as Lommel's formula [23, p. 140]. Using formula (12.5), which is the special case of (13.1) in which $\beta = 0$, we get

\[
\sqrt{\frac{2w}{\pi}} \frac{\cos\left( \sqrt{\frac{2}{w} + 2wt} \right)}{\sqrt{w^2 + 2wt}} =
\]

\[
\sum_{n=\infty}^{\infty} \frac{J_{\gamma-n-\frac{1}{2}}(w) t^{\gamma+n}}{\Gamma(\gamma+n+1)}.
\]

(13.2)
The integral of (13.2) with respect to \( t \) can be found in [1, vol. 2, p. 100]. Thus we have shown that Lommel's formula and (13.2) are related as special cases of the more general result (13.1).

**Example 13.2**

A glance at Table 11 reveals the similarity in the derivative representations for the modified Bessel \( I_\nu \), the Struve \( H_\nu \), and the modified Struve \( L_\nu \) functions with the Bessel function \( J_\nu \). It is therefore clear that the symbol \( J \) can be replaced by \( I, H \), and \( L \) in (13.1) to obtain three new relations. Similarly the pair of symbols \{ \cos, J \} in (13.2) can be replaced by the pairs \{ \cosh, I \}, \{ \sin, H \}, and \{ \sinh, L \}.

**Example 13.3**

Using the representation for the Laguerre function \( L_\nu^\alpha(z) \) from Table I with (12.4) we obtain

\[
\frac{t^{\beta-\gamma}(1+t)^{-\beta+\gamma-1} L_{-\nu}^{-\beta-\gamma-1}(w(1+t))}{\Gamma(-\beta)} = \sum_{n=-\infty}^{\infty} L_{-\nu}^{-\gamma+\nu+n+1}(w) t^n \frac{1}{\Gamma(\nu-\gamma+n+1) \Gamma(\gamma-\beta+n+1)}
\]

(13.3)
Using (12.5) we get

\[
\frac{t^{-\gamma}(1+t)^{-\nu-1} \exp(w(1+t))}{\Gamma(\nu+1) \Gamma(-\nu)} =
\]

(13.4)

\[
\sum_{n=-\infty}^{\infty} \frac{L_{\nu}^{-\gamma+n+1}(w) t^n}{\Gamma(\nu-\nu+n+1) \Gamma(\nu+n+1)}
\]

Example §3.4

The second fractional derivative representation for the Laguerre function \( L_{\nu+b}^{a-b}(z) \) from Table §3.1 in conjunction with (12.4) gives

\[
t^{\nu-\gamma} \Gamma(\beta+1) L_{\beta}^{a-b+p} (w(1+t)) =
\]

(13.5)

\[
\sum_{n=-\infty}^{\infty} \frac{\Gamma(\nu+n+1) L_{\nu+n}^{a+b-\gamma} (w) t^n}{\Gamma(\nu+b+n+1)}
\]

Using (12.5) we get
(13.6) \[ t^{-\gamma} (1+t)^{a+p} \exp(-wt) = \sum_{n=-\infty}^{\infty} I_{\gamma+n}^{a+p-\gamma-n}(w) t^n \]

The special case of (13.6) in which \( \gamma = 0 \) can be found in [2, vol.2, p. 189] and is due to Erdelyi.

Many other examples can be obtained with the aid of a table of fractional derivatives or integrals such as that found in [3, vol.2, pp. 185-214].
<table>
<thead>
<tr>
<th>Name</th>
<th>Derivative representation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Bessel function</strong></td>
<td>$J_\alpha(z) = (2z)^{-\alpha} \pi^{-\frac{1}{2}} D_{z^2}^{\frac{1}{2} - \alpha} \cos(z)$</td>
</tr>
<tr>
<td><strong>2. Modified Bessel function</strong></td>
<td>$I_\alpha(z) = (2z)^{-\alpha} \pi^{-\frac{1}{2}} D_{z^2}^{\frac{1}{2} - \alpha} \cosh(z)$</td>
</tr>
<tr>
<td><strong>3. Struve function</strong></td>
<td>$H_\alpha(z) = (2z)^{-\alpha} \pi^{-\frac{1}{2}} D_{z^2}^{\frac{1}{2} - \alpha} \sin(z)$</td>
</tr>
<tr>
<td><strong>4. Modified Struve function</strong></td>
<td>$L_\alpha(z) = (2z)^{-\alpha} \pi^{-\frac{1}{2}} D_{z^2}^{\frac{1}{2} - \alpha} \sinh(z)$</td>
</tr>
<tr>
<td><strong>5. Laguerre function</strong></td>
<td>$L_\nu(z) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1) \Gamma(-\nu)} D_{z}^{\nu - \nu - 1} \frac{e^z}{z^{\nu + 1}}$</td>
</tr>
<tr>
<td><strong>6.</strong></td>
<td>$L_{\nu + b \alpha}(z) = \frac{e^{z} z^{a+b \alpha}}{\Gamma(p+b \alpha+1)} D_{z}^{p+b \alpha} e^{z}$</td>
</tr>
<tr>
<td><strong>7. Hyper-geometric function</strong></td>
<td>$F(a,b;\nu;z) = \frac{\Gamma(\nu + 1) z^{1-\alpha}}{\Gamma(b) D_{z}^{\nu - \alpha - b - 1} (1-z)^a}$</td>
</tr>
<tr>
<td><strong>8. Incomplete Gamma function</strong></td>
<td>$\gamma(\alpha, z) = \Gamma(\alpha) e^{-z} D_{z}^{\alpha} e^{z}$</td>
</tr>
</tbody>
</table>
REFERENCES


5. ----------, Axially symmetric potentials and fractional integration, Ibid., 13 (1965), pp. 216-228.


Note: The author is indebted to T. P. Higgins of Boeing, L. A. Rubel of the University of Illinois, M. Gaer of the University of Delaware, and the editors of SIAM Publications W. S. Loud and F. W. J. Olver, and the SIAM referees for many references on fractional derivatives.
Errata for Osler's Doctoral Thesis

Leibniz rule, the chain rule and Taylor's series for fractional derivatives

P. 28  line 6 from bottom  change ≤ to ≥

p. 59  line 3 from top  \( z^q \) should be \( z^{q+p+1} \)

P. 59  line 2 from top  insert " , for \( p = 1, 2, 3, \ldots \) "

p. 59  line 4 from bottom  insert " for \( \text{Re}(\alpha) < 0 \)."

p. 60  line 3 from bottom  put \( r! \) in the denominator